Math 1c	TA: Padraic Bartlett

Recitation 6:Line Integrals

Week 6 *Caltech 2011*

1 Random Question

This problem is from my candidacy talk! It's something which neither I nor my advisor currently know the answer to, but should be eminently solvable! We think.

A **latin square** is a $n \times n$ matrix populated by the symbols $\{1, \ldots n\}$, such that no symbol is repeated in any row or column. For example, the matrix below is a 4×4 latin square:

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

A 2×2 **subsquare** of a latin square L is... perhaps best defined by an example. Consider the matrix below:

$$\begin{array}{|c|c|c|c|}\hline \boxed{1} & 2 & \boxed{3} & 4 \\ \hline 4 & 1 & 2 & 3 \\ \hline \boxed{3} & 4 & \boxed{1} & 2 \\ \hline 2 & 3 & 4 & 1 \\ \hline \end{array}$$

In this 4×4 latin square, the four boxed entries form a 2×2 subsquare. In general, a 2×2 subsquare of some latin square L is a pair of rows i, k and columns j, l in L, such that if you look at the four cells $L(i, j), L(i, l), L(k, j), L(k, l)$, you have a 2×2 latin square formed by these four cells! In other words, you have $L(i, j) = L(k, l)$ and $L(i, l) = L(k, j)$.

A question I'm trying to find the answer to is the following: can you find a $n \times n$ latin square such that every cell is involved in a "lot" of 2×2 subsquares? Maybe each cell could be involved in $n/2$ 2×2 subsquares? Or maybe just $n/16$? n over any constant?

The motivation for this question is that (1) it's really easy to find $n \times n$ latin squares on an **even** number of symbols that have this property. An example of such a square is the following:

$$L = \begin{array}{|c|c|c|c||c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 \\ \hline 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ \hline 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \\ \hline\hline 5 & 8 & 7 & 6 & 1 & 4 & 3 & 2 \\ \hline 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ \hline 7 & 6 & 5 & 8 & 3 & 2 & 1 & 4 \\ \hline 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline \end{array}$$

For a warm-up exercise: how can you generalize the above to a $2k \times 2k$ latin square with every cell involved in $n/2$ many different 2×2 subsquares?

2 Line Integrals: The Basics

Today's lecture is pretty computational and straightforward in nature! In compensation, there's a decent breadth of material to cover. So, we'll work a lot of examples here; none are too awful, as for the most part line integral calculations are relatively straightforward things. We start with a few basic definitions:

Definition. For a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a continuous path $\gamma : [a, b] \to \mathbb{R}^n$, we define the **line integral** of f along γ as

$$\int_\gamma f \cdot d\gamma := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

In the definition above, it looks like our integral $\int_\gamma f$ depends intimately on the function γ, which seems kind of weird: if we're doing a line integral, shouldn't our integral only care about the line *drawn* by the function γ, rather than the function itself?

In other words: suppose we have two paths $\gamma : [a, b] \to \mathbb{R}^n$ and $\alpha : [c, d] \to \mathbb{R}^n$, such that α and γ draw the same curve in $\mathbb{R}^n$ with the same orientation (i.e. they both start in the same place, and if the curve is a closed curve, go in the "same direction," i.e. clockwise or counterclockwise, around the curve.) Will the integrals be the same?

As it turns out, yes! In fact, we have the following theorem:

Theorem 1 *Take any two curves $\gamma : [a, b] \to \mathbb{R}^n$ and $\alpha : [c, d] \to \mathbb{R}^n$, such that α and γ have the same image in $\mathbb{R}^n$, $\alpha(a) = \gamma(c)$, and both curves traverse their images with the same orientation. Then, for any function $f : \mathbb{R}^n \to \mathbb{R}^n$, we have that*

$$\int_\gamma f \cdot d\gamma \int_\alpha f \cdot d\alpha$$

if either of these integrals exist.

So, in other words, the only thing that matters for integrating along a curve $c \subset \mathbb{R}^n$ is the curve itself – not its parametrization! Therefore, we will often just write

$$\int_c f dc$$

to denote the integral of f along the curve c. In the case where c is a closed curve, we will usually indicate its orientation in the definition of c.

To illustrate the basics here, we do two quick examples:

Example. For the function

$$f(x, y) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right),$$

what is the integral of f around the circle c_r of radius r traversed in the counter-clockwise direction?

Solution. First, by our theorem above, we know that we can use any counter-clockwise parametrization of our circle to find this integral. The easiest one to use is the standard parametrization of the circle,

$$\gamma(t) = (r\cos(t), r\sin(t)), t \in [0, 2\pi].$$

(As an aside: by using polar coördinates, it's trivially clear that this function graphs out the circle of radius r; furthermore, because initially increasing t moves us up and to the left, it is clear that this is a counterclockwise parametrization of said circle.)

Therefore, we can just use our theorem, which says that

$$\begin{aligned}
\int_{c_r} f \cdot dc &= \int_0^{2\pi} \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right) \Big|_{(r\cos(t), r\sin(t))} \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(\frac{2r\cos(t)}{r^2\cos^2(t)+r^2\sin^2(t)}, \frac{2r\sin(t)}{r^2\cos^2(t)+r^2\sin^2(t)} \right) \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(\frac{2r\cos(t)}{r^2}, \frac{2r\sin(t)}{r^2} \right) \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(-\frac{2r^2\cos(t)\sin(t)}{r^2} + \frac{2r^2\sin(t)\cos(t)}{r^2} \right) dt \\
&= \int_0^{2\pi} 0 dt \\
&= 0
\end{aligned}$$

Example. Show that the two paths

$$\begin{aligned}
\gamma(t) &= (t, t), t \in [0, 2] \\
\alpha(t) &= \left(t(2t-3)^2, t(2t-3)^2\right), t \in [0, 2]
\end{aligned}$$

trace out the same path with the same orientation.

Solution. So: notice that

- $t(2t-3)^2\Big|_0 = 0$,
- $t(2t-3)^2\Big|_1 = 1$,
- $t(2t-3)^2$'s only other root is at $t = 3/2$, and
- $t(2t-3)^2\Big|_2 = 2(2 \cdot 2 - 3)^2 = 2$.

Therefore, $t(2t-3)^2$ is a continuous function that's 0 at $t = 0$, 2 at $t = 2$, and $\geq$ zero on the entire interval $[0, 2]$. (This last step is justified by the intermediate value theorem, our observation that f has its only zeroes at 0 and 3/2, and it takes on a positive value on either side of 3/2.)

Therefore, the graph of $\alpha(t) = \big(t(2t-3)^2, t(2t-3)^2\big)$ on $[0,2]$ will be the same as $\gamma(t)$! This is because it will hit every point of the form (x,x) for $x \in [0,2]$ – in fact, it will hit some of these points multiple times! – and will hit only points of this form, as we've established with our discussion above.

As a somewhat surprising corollary, this means that integrating along either of these paths is the same, even though integrating along α involves going out from $(0,0)$ to $(1,1)$, then back to $(0,0)$, then back again to $(2,2)$.

3 Line Integrals w/r/t Arc Length

I didn't mention this in recitation, but it bears noting that we can also integrate a **scalar field** over a curve as well! We do this with the **line integral with respect to arc length**, defined here:

Definition. Similarly, suppose now that we have a function $f : \mathbb{R}^n \to \mathbb{R}$ – i.e. a scalar field instead of a vector field – and a continuous path $\gamma : [a,b] \to \mathbb{R}^n$. Then, we can define the **line integral with respect to arc length** of f along γ as

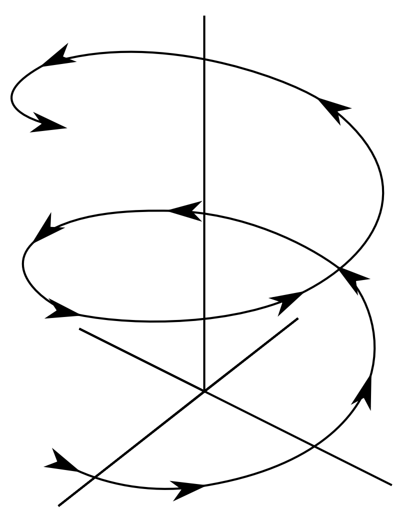
$$\int_\gamma f \cdot d\gamma := \int_a^b f(\gamma(t)) \cdot ||\gamma'(t)|| dt.$$

Just like with the normal line integral, this only depends on the curve drawn by γ and not the specific parametrization given by γ itself.

To illustrate this definition, we calculate a quick example:

Example. Integrate the function $f(x,y,z) = x^2y^2 + y^2z^2 + x^2z^2$ over the helix $\gamma(t) = (\cos(t), \sin(t), t), t \in [0, 2\pi)$.

Solution. Because it's pretty, we sketch this curve here:



Ok! Now, with that done, we just simply apply our above definition:

$$\begin{aligned}
\int_\gamma f(x,y,z)\cdot d\gamma &= \int_0^{2\pi} f(\cos(t),\sin(t),t)\cdot ||(\cos(t),\sin(t),t)'||dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t) + t^2\sin^2(t) + t^2\cos^2(t))\cdot ||(-\sin(t),\cos(t),1)||dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t) + t^2)\cdot \sqrt{\sin^2(t)+\cos^2(t)+1^2}dt \\
&= \int_0^{2\pi} \left(\frac{\sin^2(2t)}{4} + t^2\right)\cdot \sqrt{2}dt \\
&= \int_0^{2\pi} \left(\frac{1-cos(4t)}{8} + t^2\right)\cdot \sqrt{2}dt \\
&= \left(\frac{t}{8} - \frac{\sin(4t)}{32} + \frac{t^3}{3}\right)\sqrt{2}\Big|_0^{2\pi} \\
&= \left(\frac{2\pi}{8} - \frac{0}{32} + \frac{(2\pi)^3}{3}\right)\sqrt{2} - 0 \\
&= \frac{2\pi\sqrt{2}}{8} + \frac{8\pi^3\sqrt{2}}{3}.
\end{aligned}$$

4 Path-Connected Sets

There are a number of more powerful theorems on line integrals; however, to develop them, we first need to introduce the concept of a **path-connected** set. We do this here, and illustrate it via a few examples:

Definition. A set $S \subset \mathbb{R}^n$ is called **path-connected** if for any two points $\mathbf{x}, \mathbf{y} \in S$, there is a continuous path $\gamma : [a,b] \to \mathbb{R}^n$, contained entirely within S, such that $\gamma(a) = \mathbf{x}$ and $\gamma(b) = \mathbf{y}$. In class, we will often simply shorten this to "connected."

(You may remember this definition from week 1's random question in recitation!)

To illustrate the use of this definition, we work two examples here; the first is relatively straightforward, while the second is a bit less so:

Example. Show that the n-dimensional sphere of radius $r > 0$,

$$S = \{\mathbf{x} : ||\mathbf{x}|| = r\}$$

is path-connected.

Solution. First, think of $\mathbb{R}^n$ in spherical coördinates, where we identify a point $\mathbf{x}$ in $\mathbb{R}^n$ with the coördinates $(r, \theta_1, \dots \theta_{n-1})$, where

- r denotes the distance of $\mathbf{x}$ from the origin, and

- each $\theta_1 \dots \theta_{n-1}$ indicate the angle made by $\mathbf{x}$ with the vector e_i, with $\theta_1, \dots \theta_{n-2} \in [0, \pi)$, $\theta_{n-1} \in [0, 2\pi)$.

Then, if you pick any two points $(r, \theta_1, \dots \theta_{n-1})$, $(r, \gamma_1, \dots \gamma_{n-1})$ on the unit circle, you can trivially move between them via the path

$$\gamma(t) = (r, \theta_1 + t \cdot (\gamma_1 - \theta_1), \dots \theta_{n-1} + t \cdot (\gamma_{n-1} - \theta_{n-1})), t \in [0, 1].$$

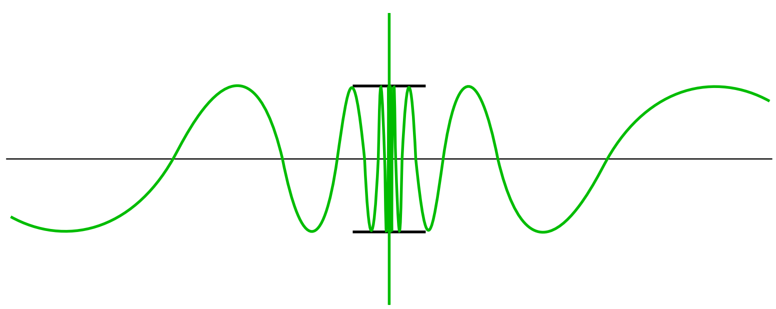
If you're disconcerted by our use of spherical coördinates and would prefer to see this map as a function in Euclidean coördinates, just compose our function γ with the map that transforms spherical coördinates to Euclidean coördinates:

- $x_1 = r\cos(\theta_1)$,
- $x_2 = r\sin(\theta_1)\cos(\phi_2)$,
- $x_3 = r\sin(\theta_1)\sin(\theta_2)\cos(\phi_3)$,
- $\dots$
- $x_{n-1} = r\sin(\theta_1)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1})$,
- $x_n = r\sin(\theta_1)\cdots\sin(\theta_{n-2})\sin(\theta_{n-1})$.

Example. Is the set

$$S := \left\{ (x, y) : (x, y) \text{ is either of the form } (0, y) \text{ or } \left(x, \sin\left(\frac{1}{x}\right)\right) \right\}$$

path-connected?



Solution. So, as it turns out, this set is not path-connected! This is perhaps surprising, as it certainly *looks* connected: visually, it all appears to be one piece, at the least. (as well, this set *is* connected, for several other definitions of the word "connected:" in week one, we mentioned that one possible definition of connected could be "a set that cannot be divided into two pieces A, B, such that each piece can be contained in an open set U, V, where these open sets don't intersect." This notion of "connected" – that you can't chop the set into two pieces that are "some distance away" from each other – *is* one that our set S preserves!)

To see why this set is not path-connected: take any point $(0, y)$ on the y-axis, and any other point $(x, \sin(1/x))$ on the curve-part of our set. Suppose that there was a path

$\gamma : [0,1] \to S$ connecting these two points; then, in order to do this, it will have to travel along all of the curve $(t, \sin(1/t))$ from t to 0. But we know that the function $y = \sin(1/x)$ is not continuous at 0, as it constantly oscillates between ± 1 as it approaches zero! Therefore, any map restricted to this function must also not be continuous if its x-coördinate approaces zero; therefore, γ cannot be continuous, and is therefore not a path!

Thus, no such path can exist, and this set is not path-connected.

5 Line Integrals and Gradients

With this definition made, we can state the following theorem:

Theorem 2 *Suppose that S is an open and path-connected set. Then, the following conditions are equivalent, for any function $f : S \to \mathbb{R}^n$:*

1. *There is a scalar field $F : S \to \mathbb{R}$ such that $\nabla F = f$.*
2. *The line integral of f over any path $\gamma : [a,b] \to S$ only depends on γ's endpoints: i.e.*
$$\int_\gamma f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a)).$$
3. *The line integral of f over any closed path $\gamma : [a,b] \to S$ (i.e. any path γ with $\gamma(a) = \gamma(b)$) is identically zero.*

As we really don't have any tools yet to talk about "all of the paths" γ in a space S, the way we will usually use this theorem is to (1) notice that a given function is a gradient, and then (2) deduce that an otherwise-difficult integral is trivially given by evaluating f on its endpoints, or is zero (because the curve is closed.)

We present two examples of this method here:

Example. Find the line integral of the vector field $f(x,y) = (x^2, y^2)$ over the curve $\gamma(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$.

Solution. If we want to do this the hard way, we can just calculate:

$$\begin{aligned}
\int_\gamma f \cdot d\gamma &= \int_0^{2\pi} f(\cos(t), \sin(t)) \cdot (-\sin(t), \cos(t)) dt \\
&= \int_0^{2\pi} (\cos^2(t), \sin^2(t)) \cdot (-\sin(t), \cos(t)) dt \\
&= \int_0^{2\pi} (-\sin(t)\cos^2(t) + \cos(t)\sin^2(t)) dt \\
&= \int_0^{2\pi} -\sin(t)\cos^2(t) dt + \int_0^{2\pi} \cos(t)\sin^2(t)) dt \\
&= \int_0^0 u^2 du + \int_0^0 u^2 du \\
&= 0,
\end{aligned}$$

where we justified the second-to-last step by making the two substitutions $u = \cos(t)$ and $u = \sin(t)$, respectively, in our two integrals.

Alternately, if we want to do this the easy way, we can notice that our function $f(x, y)$ is the gradient of the function $F(x, y) = \frac{x^3+y^3}{3}$. (We don't have any terribly good methods yet in this course for "finding" these gradients; mostly, it's just a process of pattern recognition and clever guessing.)

In the above situation, we had a problem where we could actually just calculate the answer, instead of using our theorem and finding f as a gradient. Sometimes, however, this is impossible, as we'll have curves that we will most emphatically **not** want to integrate:

Example. Find the line integral of the vector field $f(x, y) = (yz, xz, xy)$ over the curve

$$\gamma(t) = (1, \cos(t), W(t)), t \in [0, 2\pi]$$

where $W(t)$ is a Weierstrass function (a function that's everywhere continuous and nowhere-differentiable),specifically defined by the infinite sum

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos\left(101^n \cdot \pi x\right)}{2^n}.$$

Solution. So: if you want to do that directly, good luck!

Otherwise, if you don't like the idea of integrating an infinite sum of cosines, we can instead simply notice that because $\cos(0) = \cos(101^n \cdot 2\pi) = 1$, we have that

$$\gamma(0) = (1, \cos(0), W(0)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = th(1, 1, 1)$$
$$\gamma(2\pi) = (1, \cos(2\pi), W(2\pi)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = (1, 1, 1)$$

and thus that this curve is closed.

Therefore, because $f(x, y, z)$ can be written as the gradient of $F(x, y, z) = xyz$, we know that this integral is zero, without doing any work at all!