Math 1c	TA: Padraic Bartlett

Recitation 7: Green's Theorem

Week 7	*Caltech 2011*

1 Random Question

You and $n-1$ of your friends have been teleported in time and space to Soviet Russia, where the KGB has seized you and forced you to play the following game:

- Setup: you and your friends are all going to be placed in a circle, and have fedoras placed on your heads. The fedoras all have numbers $\{1, \ldots n\}$ written in cards tucked into their brims; you cannot see the card in the hat on your own head, but can see everyone else's card and hat. The numbers may be repeated; it is possible that all of the hats may be labeled 2, or perhaps 1,2,3,1,2,3..., or any other sequence of numbers from the set $\{1, \ldots n\}$.
- Game: Once the hats have been brought out and placed on your head, none of you are allowed to speak or communicate in any way with one another, as enforced by a collection of paranoid KGB agents.
- Once you've all settled into your hats, a guard will count to three; at the count of three, you all must shout out which number you believe to be on your hat.
- You win if at least one person guesses the right number on their hat, and are sent back to the modern day and age! Otherwise, you lose, and are sent to Siberia for various and sundry acts of toil for the rest of your lives.

You're all told these rules ahead of time, and given a day to prepare a strategy. Can you insure that you'll always make it back to 2011?

2 Green's Theorem: Statement and Definitions

This lecture is on Green's theorem, a remarkably powerful result that tells us how to transform line integrals into integrals over an entire region. We open here with the statement of the theorem; from there, we'll calculate three distinct examples, which will help illustrate the three main ways that Green's theorem is typically used.

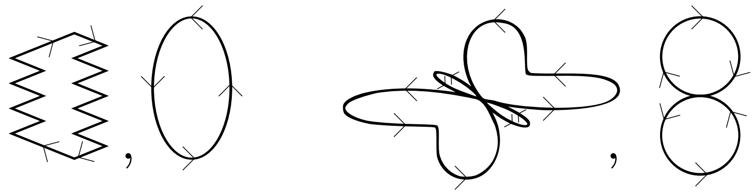
Before we can say what Green's theorem, though, we should remember the definition of a **simple closed curve**, which we restate below:

Definition. A **simple closed curve** γ is a map $[a, b] \to \mathbb{R}^n$ such that

- $\gamma(a) = \gamma(b)$,
- γ has finite length, and

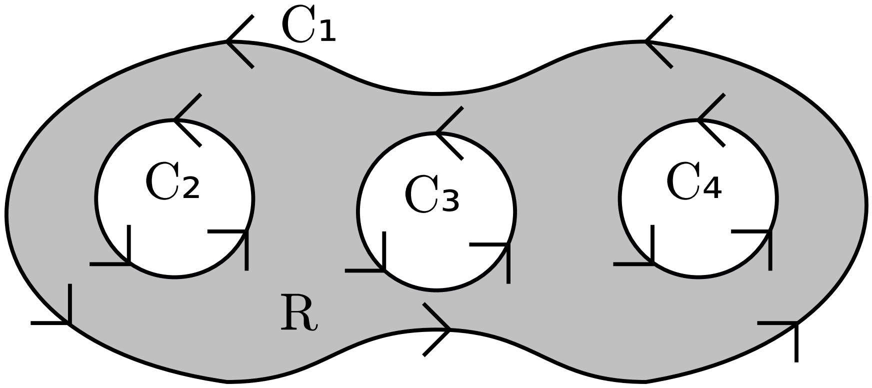
- γ does not intersect itself: i.e. for any two points $x \neq y \in [a, b]$, $\gamma(x) = \gamma(y)$ if and only if x and y are the two endpoints a, b.

Example. The following illustrates some closed curves that are simple, and some closed curves that are not simple:



(simple closed curves) (not simple closed curves)

Theorem 1 *(Green's Theorem, general case) Suppose that R is some closed, bounded, path-connected region in $\mathbb{R}^2$, such that R's boundary is formed by the curves $C_1, \ldots C_n$, where C_1 goes around the outside of R, the curves $C_2, \ldots C_n$ all lie within C_1, and all of these curves are oriented in the counterclockwise direction. In other words, suppose that we are in the situation depicted in the picture below:*



Now, suppose that P and Q are a pair of maps $\mathbb{R}^2 \to \mathbb{R}$ with continuous partial derivatives in an open neighborhood of R. Then, we have the following equality:

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \left(\oint_{C_1} Pdx + Qdy \right) - \sum_{i=2}^{n} \left(\oint_{C_i} Pdx + Qdy \right)$$

Often, R will wind up being a region with no holes in it, like the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$ or the unit square $[0, 1]^2$; in this case, we have the more traditional form of Green's theorem, which simply says

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{C_1} Pdx + Qdy$$

3 Green's Theorem: Three Applications

Why do we care about Green's theorem? Well: from looking at its statement above, what does it do? It takes a pair of functions P, Q and sends an integral involving them to an integral involving their partials $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$; as well, it transforms a line integral over some curve C into a integral over some region R. This suggests that we might want to use Green's theorem in the following situations:

1. If we're integrating a pair of functions over some particularly awful curve, we might want to use Green's theorem to transform this integral into one over a region, in the hopes that the expression $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ might become zero or at the least a simpler expression.

2. Conversely, if we have a fairly awful region R, we might want to use Green's theorem to take us to a line integral, which can sometimes make our lives easier. One typical example of this is the use of Green's theorem to calculate the **area** of a region, which is the following equation:

$$\iint_R 1 \; dxdy = \oint_C xdy - ydx.$$

   The left-hand side is (by definition) the area of the region R; the right-hand side is one possible pair of functions P, Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is 1.

3. The above two situations allow us to switch a line integral for an area integral, or vice-versa. However, it bears noting that in certain special situations, we can use Green's theorem to switch between a pair of line integrals! Specifically: suppose that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is identically zero on some region R. Then, for any two simple closed curves C_1, C_2 inside of R, where C_1 is contained inside of C_2, Green's theorem says the following:

$$0 = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \left(\oint_{C_1} Pdx + Qdy \right) - \left(\oint_{C_2} Pdx + Qdy \right)$$
$$\Rightarrow \left(\oint_{C_1} Pdx + Qdy \right) = \left(\oint_{C_2} Pdx + Qdy \right).$$

   So, in this situation, Green's theorem allows us to switch our integral between these two curves! This can be remarkably useful, as some curves (like circles) are much easier to integrate along than others (like n-gons.)

We illustrate these three uses with three examples:

Example. For any two constants $a, b \in \mathbb{R}$, and $n \in \mathbb{N}$, find the integral

$$\oint_{C_n^+} adx + bdy,$$

where C_n^+ is a counterclockwise-oriented n-gon with side length 1, center at (0,0), and one vertex on the x-axis.

Solution. So: this is (clearly) a case where our curve C_n^+ is far too awful to integrate along. Having no other option, we apply Green's theorem, which tells us that (if R is the region enclosed by our n-gon)

$$\begin{aligned}\oint_{C_n^+} adx + bdy &= \iint\limits_R \left(\frac{\partial(b)}{\partial x} - \frac{\partial(a)}{\partial y}\right) dxdy \\ &= \iint\limits_R (0-0)\, dxdy \\ &= 0.\end{aligned}$$

(As a side note, notice that this trivially holds for any closed curve! Therefore, by the theorem relating gradients and line integrals we discussed last week, the function $(x, y) \mapsto (a, b)$ has a potential – i.e. it can be written as the gradient of some function! In specific, one such function is $F(x, y) = ax + by$.)

Example. Find the area of the ellipse

$$R = \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}.$$

Solution. As mentioned before, the area of any region R can be given by the integral $\iint\limits_R 1\ dxdy$; so, if we choose $P(x, y) = -y/2, Q(x, y) = x/2$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, and thus that

$$\iint\limits_R 1\ dxdy = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \frac{1}{2}\oint_{C^+} xdy - ydx,$$

where C^+ is the boundary curve of our ellipse: i.e. $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $\gamma(t) = (a\cos(t), b\sin(t))$.

Calculating, we have

$$\begin{aligned}\frac{1}{2}\oint_{C^+} xdy - ydx &= \frac{1}{2}\int_0^{2\pi} (-y, x)\Big|_{\gamma(t)} \cdot \gamma'(t)dt \\ &= \frac{1}{2}\int_0^{2\pi} (-b\sin(t), a\cos(t)) \cdot (-a\sin(t), b\cos(t))dt \\ &= \frac{1}{2}\int_0^{2\pi} ab(\sin^2(t) + \cos^2(t))dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\ dt \\ &= ab\pi.\end{aligned}$$

It bears noting that we had many possible choices of P, Q above! Specifically, we could have also chosen $Q = x, P = 0$; in this case, we would have had

$$\begin{aligned}
\iint_R 1 \, dxdy &= \oint_{C^+} xdy \\
&= \int_0^{2\pi} (0, a\cos(t)) \cdot (-a\sin(t), b\cos(t))dt \\
&= \int_0^{2\pi} ab\cos^2(t)dt \\
&= \int_0^{2\pi} ab\frac{\cos(2t)+1}{2}dt \\
&= \left(ab\frac{\sin(2t)}{4} + \frac{abt}{2} \right) \Big|_0^{2\pi} \\
&= ab\pi.
\end{aligned}$$

This is the same answer! This is just an aside, to illustrate that you can have many different choices of P, Q available to you such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is equal to your desired expression.

Example. Let $f(x, y)$ be the following function from $\mathbb{R}^2 \to \mathbb{R}$:

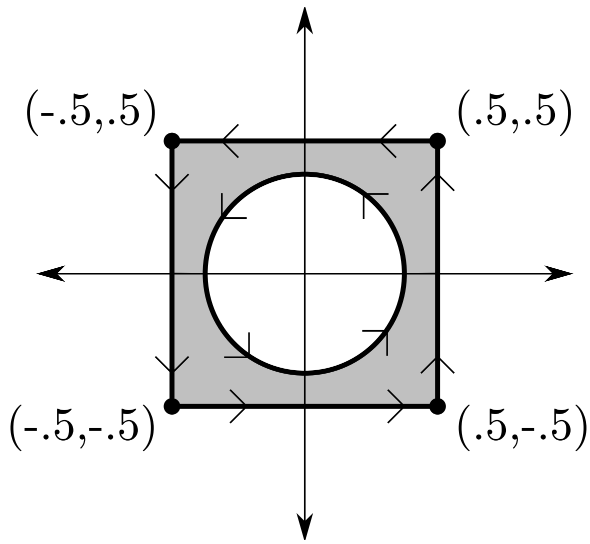
$$f(x,y) = \begin{cases} \arctan(y/x), & x > 0, \\ \pi/2, & x = 0, y > 0, \\ -\pi/2, & x = 0, y < 0, \\ \arctan(y/x) + \pi, & x < 0, y > 0, \\ \arctan(y/x) - \pi, & x < 0, y < 0, \end{cases}$$

and let $F(x, y) = \nabla f(x, y)$. On the HW last week, we showed that on $\mathbb{R}^2 \setminus \{(x, 0) : x < 0\}$, we had

$$F(x,y) = \nabla f(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

Take this function $F(x, y)$ and extend it to all of $\mathbb{R}^2 \setminus \{(0, 0)\}$, as it's clearly defined at every point that's not the origin.

If C_1^+ is the counterclockwise curve that corresponds to a square of side length 1 centered at the origin, with sides parallel to the x- and y-axes, what's the integral of $F(x, y)$ over C_1^+?



Solution. To find this, we *could* just integrate around this square; but it would be a fairly ugly calculation. Instead, what we can do is notice that (if we think of $F(x,y)$ as $(P(x,y), Q(x,y))$, where $P(x,y) = -\frac{y}{x^2+y^2}$ and $Q(x,y) = \frac{x}{x^2+y^2}$) we have

$$\begin{aligned}
\frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{(x^2+y^2)-2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}, \text{ and} \\
\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} - \frac{y}{x^2+y^2} = -\frac{(x^2+y^2)-2y^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2} \\
\Rightarrow \frac{\partial Q}{\partial x} &- \frac{\partial P}{\partial y} = 0.
\end{aligned}$$

Therefore, we know that on any region R that doesn't contain a small ball around the origin, we have $\iint\limits_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = 0$. Consequently, we can use our curve-switching technique we discussed earlier, to see that for any curve C_2^+ contained entirely within our square C_1^+, we have $\oint_{C_1^+} Pdx + Qdy = \oint_{C_2^+} Pdx + Qdy$. So: because it seems like it will be a far easier curve to work with, take the circle of radius $1/3$, parametrized by $\gamma(t) = \left(\frac{1}{3}\cos(t), \frac{1}{3}\sin(t)\right)$. Then, we have

$$\begin{aligned}
\oint_{C_1^+} Pdx + Qdy = &\oint_{C_2^+} Pdx + Qdy \\
&\oint_0^{2\pi} \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\Bigg|_{\gamma(t)} \cdot \gamma'(t)dt \\
&\oint_0^{2\pi} \left(-\frac{\frac{1}{3}\sin(t)}{\frac{1}{9}\sin^2(t)+\frac{1}{9}\cos^2(t)}, \frac{\frac{1}{3}\cos(t)}{\frac{1}{9}\sin^2(t)+\frac{1}{9}\cos^2(t)}\right) \cdot \left(-\frac{1}{3}\sin(t), \frac{1}{3}\cos(t)\right) dt \\
&\oint_0^{2\pi} 1dt = 2\pi.
\end{aligned}$$

4 Supplemental: Grad, Div, and Curl

Also appearing on your HW are the divergence and curl operators. We aren't really doing anything with these things other than defining them this week, so don't worry too much about them; but, for the sake of completeness, we define these operators here and calculate a few examples.

Definition. For a differentiable function $F : \mathbb{R}^3 \to \mathbb{R}^3$, we defined the **divergence** of F, often denoted either as $\text{div}(F)$ or $\nabla \cdot F$, as the following function $\mathbb{R}^3 \to \mathbb{R}$:

$$\text{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

As well, for such a function F, we define the **curl** of F, denoted $\text{curl}(F)$ or $\nabla \times F$, as the following map $\mathbb{R}^3 \to \mathbb{R}^3$:

$$\text{curl}(F) = \nabla \times F = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right).$$

Often, the curl is written as the "determinant" of the following matrix:

$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Example. Take any C^2 function $F : \mathbb{R}^3 \to \mathbb{R}^3$. What's the divergence of the curl of F?

Solution. So, by definition, we have that

$$\text{curl}(F) = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right)$$

and thus that

$$\begin{aligned} \text{div}(\text{curl}(F)) &= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} \right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} \right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \right) \\ &= \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y} \right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z} \right) + \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x} \right) \\ &= 0. \end{aligned}$$

We justify this last step by noting that F is C^2, and thus when we take mixed partials of F we don't care in which order they're evaluated.

Similarly, you can show that the curl of any gradient is zero; the proof is pretty much the same.