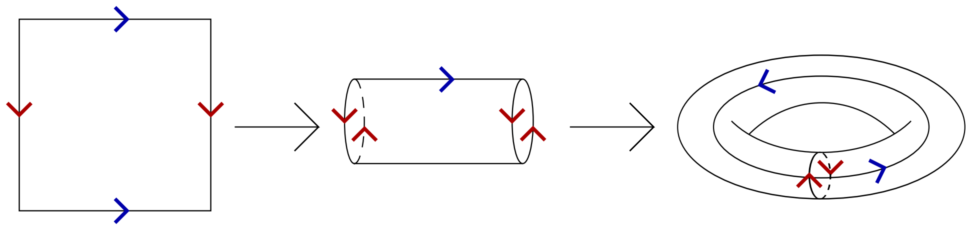
Math 1c — TA: Padraic Bartlett

Recitation 9: Integrals on Surfaces; Stokes' Theorem

Week 9 — *Caltech 2011*

1 Random Question

In the diagram below, we illustrate how to "glue together" the sides of a square to create a torus:



Basically, what we've done here is taken the two blue-arrow edges and stuck them together so that when they're pressed together, their arrows are pointing in the same direction; then, we did the same thing with the two red-arrow edges. (Try this with some paper to see what's going on!)

So: what other shapes can you get? Specifically:

1. What other shapes can you make out of the square, by identifying edges in different ways? Can you make the Klein bottle? Möbius strip[1]? Sphere?
2. How about a hexagon? Can you make a torus out of that? Any new shapes?
3. Suppose you use a $4k$-gon, instead of just a square. Can you make a k-hole torus – that is to say, a surface with k holes in it?

2 Surfaces

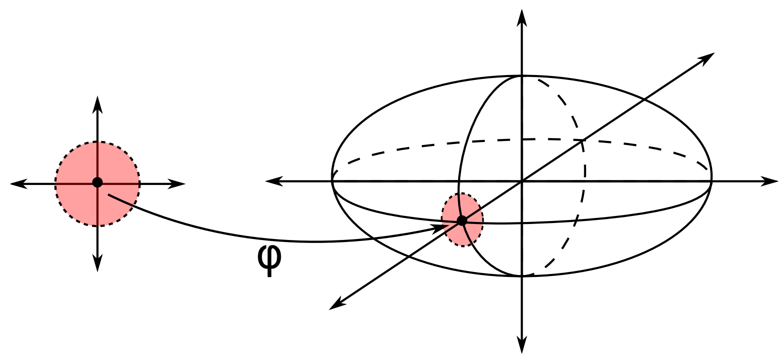
2.1 Basic definitions.

What *is* a surface? Well, to get an idea of what we might want one to be, let's examine several things we **want** to be surfaces: spheres, tori, cones, paraboloids, sheets, cubes, and the graphs of continuous functions $z = f(x, y)$. What do these all have in common? Well, intuitively speaking, they all look like they are in some sense locally *flat* – i.e. they look like they're made out of pieces of $\mathbb{R}^2$. For example, if you pick a point on a sphere, say, and zoom in really close, it basically looks identical to $\mathbb{R}^2$ there, and the same holds for the rest of our surfaces.

As it turns out, this notion, of "locally looking like $\mathbb{R}^2$," is an excellent candidate for the definition of a surface. In the following definition, we make this notion rigorous:

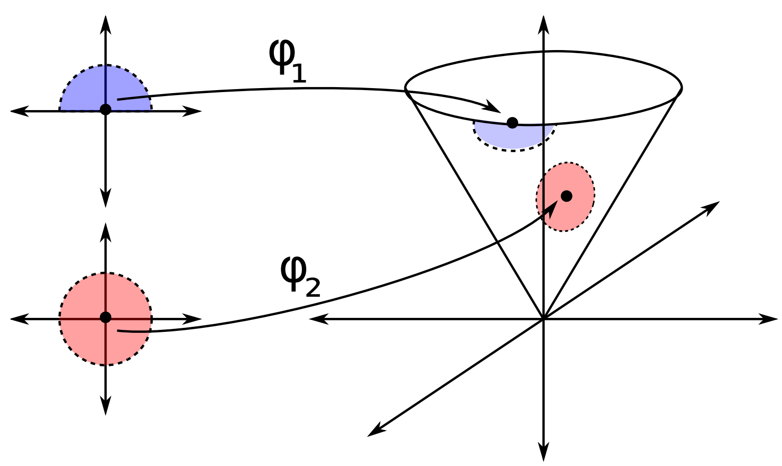
[1] If you haven't heard of these shapes before, go read about them on Wikipedia! They're quite cool.

Definition. A subset $S \subseteq \mathbb{R}^n$ is called a **surface without boundary** if for every point $s \in S$, there is an open neighborhood N_s of s and a continuous, 1-1 and onto function φ from the open unit disk $\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ to the set $N_s \cap S$. In other words, for every point s in S, there is a little neighborhood of s in which S locally looks like $\mathbb{R}^2$.



Similarly, a surface $S \subseteq \mathbb{R}^n$ is called a **surface with boundary** if for every point $s \in S$, we have one of the following two cases:

1. There is an open neighborhood N_s of s and a continuous, 1-1 and onto function $\varphi_1 : \{(x, y) : x^2 + y^2 < 1, y \geq 0\} \to N_s$. In this case, s is a boundary point of S.
2. There is an open neighborhood N_s of s and a continuous, 1-1 and onto function $\varphi_2 : \mathbb{D} \to N_s$. In this case, s is an interior point of S.

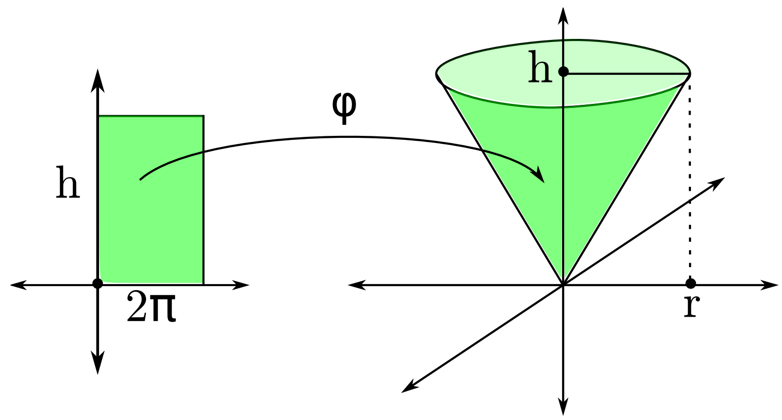


One, somewhat frustrating thing about this definition is that it only gives us these maps φ locally. However, there are many situations where we'd like to have just **one** map φ that makes everything on our surface S look like $\mathbb{R}^2$, instead of a ton of different little maps for every point in S! This motivates the following definition:

Definition. We say that S is a **surface parametrized by** φ iff there is a path-connected region $R \subset \mathbb{R}^2$ and continuous onto function $\varphi : R \to S$, that's one-to-one except perhaps on the boundary points of R.

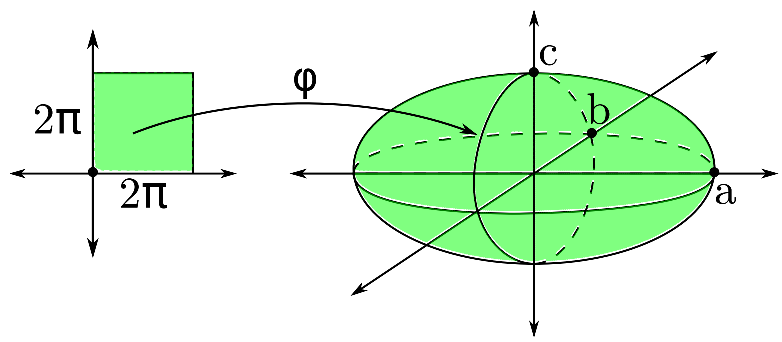
This definition is perhaps best illustrated by a series of examples of parametrized surfaces:

Example. A cone of height h and radius r around the z-axis, as depicted below, can be parametrized by the map $\varphi : [0, h] \times [0, 2\pi] \to \mathbb{R}^3$, $\varphi(z, \theta) = \left(\frac{z \cdot r}{h} \cos(\theta), \frac{z \cdot r}{h} \sin(\theta), z\right)$.



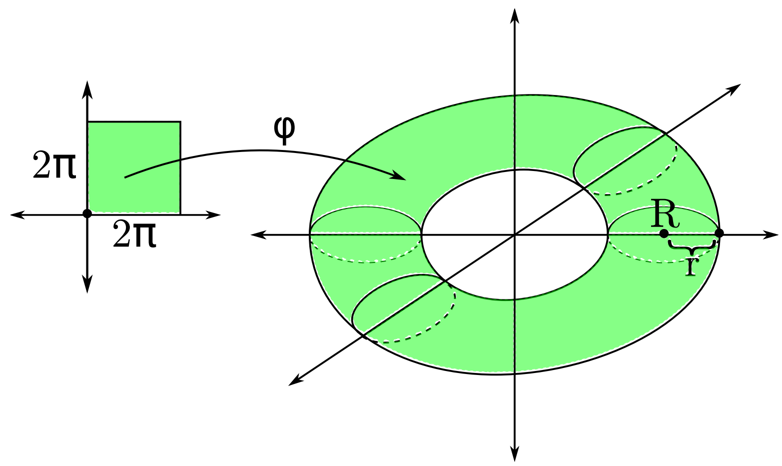
If you want to double-check this, simply use cylindrical coördinates to see that the image of the set above is indeed a cone!

Example. A ellipsoid that intersects the x-axis at a, y-axis at b, and z-axis at c, as depicted below, can be parametrized by the map $\varphi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$, $\varphi(\phi, \theta) = (a \sin(\phi) \cos(\theta), b \sin(\phi) \sin(\theta), c \cos(\phi))$.

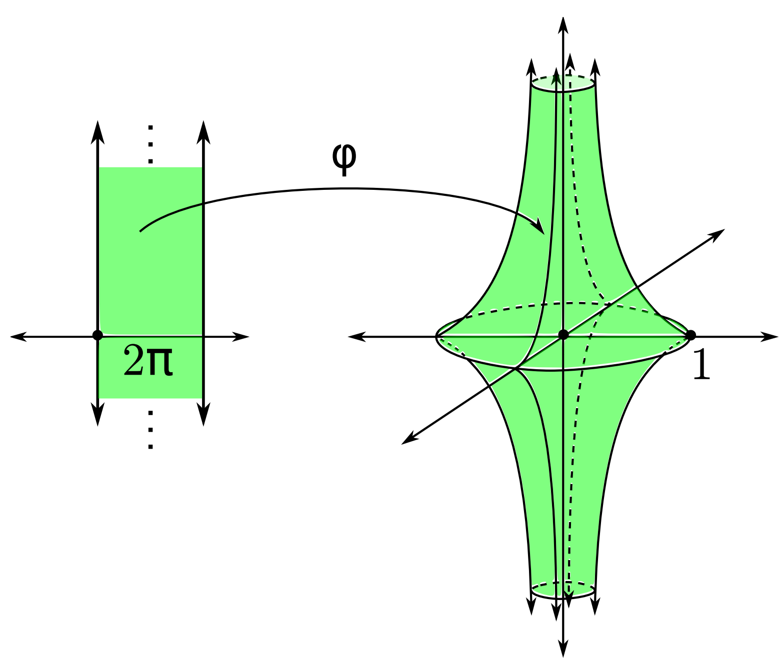


Similarly to the above, you can double-check that this is valid by using spherical coördinates.

Example. A torus around the circle $x^2 + y^2 = R^2$, with internal radius r (as depicted below) can be parametrized by the map $\varphi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$, with
$\varphi(\phi, \theta) = (\cos(\phi)(R + r\cos(\theta)), \sin(\phi)(R + r\cos(\theta)), r\sin(\theta))$.



Example. The pseudosphere is a strange surface, that in several senses can be thought of as a natural complement to the idea of a sphere. In a sense we will define in future Caltech math courses, it has curvature -1 everywhere (as opposed to the sphere, which has curvature 1 everywhere;) as well, it has the same surface area ($4\pi r^2$) as a sphere, and half of the volume ($2\pi r^3/3$) of the sphere. It is parametrized by the map $\varphi : (-\infty, \infty) \times [0, 2\pi] \to \mathbb{R}^3$, $\varphi(t, \theta) = (\mathrm{sech}(t)\cos(\theta), \mathrm{sech}(t)\sin(\theta), t - \tanh(t))$, where the expressions sech and tanh are the **hyperbolic trig functions** $\mathrm{sech}(t) = \frac{2}{e^t + e^{-t}}$, $\tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$.



2.2 Integrals on surfaces.

As this is a calculus class, the natural question to ask (when given any new object) is "How can we take integrals with this?" In other words, suppose we have a surface $S \subset \mathbb{R}^3$, and some function $f : \mathbb{R}^3 \to \mathbb{R}$. What would we possibly mean by the **integral** of f on S?

Well: suppose for the moment that f is parametrized by some function $\varphi : R \to S$, $R \subseteq \mathbb{R}^2$. Then, one natural way to define the integral of f over S is to say that it is the integral of $f \circ \varphi$ over R, where we need to compensate for how φ "stretches areas." To be explicit, we have the following definition:

Definition. For a surface $S \subset \mathbb{R}^3$ parametrized by some function $\varphi(x, y) : R \to S$, $R \subseteq \mathbb{R}^2$ and some function $f : \mathbb{R}^3 \to \mathbb{R}$, we define the **integral** of f over S as the following expression:

$$\iint\limits_S f \; dS = \iint\limits_R f(\varphi(x, y)) \cdot \left|\left| \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} \right|\right| dxdy.$$

The $\left|\left| \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} \right|\right|$ bit above, specifically, is the thing that corrects for how φ distorts space. Specifically, at any point (x, y), it's distorting space by $\frac{\partial \varphi}{\partial x}$ as you increase x slightly and by $\frac{\partial \varphi}{\partial y}$ as you increase y: therefore, it's distorting area by the magnitude of the cross-product of those two vectors at that point!

2.3 Example calculations.

To demonstrate how these concepts work, we calculate two examples:

Example. Calculate the surface area of a cone C with height 1 and radius 1 (using the height and radius notation from our earlier parametrizations.)

Solution. (First, note that the surface area of any surface S is just the integral of the function 1 over the entire surface: therefore, this problem is just asking us to find $\iint\limits_S 1 \; dS$.

Let $\varphi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$, $\varphi(z, \theta) = (z \cos(\theta), z \sin(\theta), z)$ be the parametrization of the cone we discussed earlier in recitation. Then, by definition, we have that

$$\begin{aligned} \iint\limits_C 1 \; dS &= \int_0^1 \int_0^{2\pi} 1 \cdot \left|\left| \frac{\partial \varphi}{\partial z} \times \frac{\partial \varphi}{\partial \theta} \right|\right| d\theta dz \\ &= \int_0^1 \int_0^{2\pi} 1 \cdot ||(\cos(\theta), \sin(\theta), 1) \times (-z \sin(\theta), z \cos(\theta), 0)|| \; d\theta dz \\ &= \int_0^1 \int_0^{2\pi} \left|\left|(-z \cos(\theta), -z \sin(\theta), z \cos^2(\theta) + z \sin^2(\theta))\right|\right| d\theta dz \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{2\pi} ||(-z\cos(\theta), -z\sin(\theta), z)||\, d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \sqrt{(-z\cos(\theta))^2 + (z\sin(\theta))^2 + z^2} d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \sqrt{z^2\cos^2(\theta) + z^2\sin^2(\theta)2 + z^2} d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \sqrt{2z^2} d\theta dz \\
&= \int_0^1 \int_0^{2\pi} z\sqrt{2} d\theta dz \\
&= \int_0^1 2\pi z\sqrt{2} dz \\
&= \pi\sqrt{2}.
\end{aligned}
$$

Example. Find the center of mass of a cone C centered on the z-axis of height 1 and radius 1, if it has uniform area density 1 (i.e. its area density function is $\gamma(x, y, z) = 1$.)

Solution. (Recall that a **center of mass** for any object is a point (x, y, z) such that any plane cutting through (x, y, z) will have half of the object's mass on either side of this plane. Also, recall that we can find this by finding the *average* of each of the coördinates x, y, z over this surface, weighted by the density function $\gamma(x, y, z)$: in other words, the x-coördinate of the center of mass of any surface with density function γ is just the quantity $\iint\limits_S x \cdot \gamma dS$ divided by the total mass of S, $\iint\limits_S \gamma dS$.

First, notice that any such cone centered on the z-axis must have the x- and y-coördinates of its center of mass both be zero, as this cone is symmetric around the x- and y-axes.

So, it suffices to find the z-coördinate of the center of mass of our cone. To do this, because our density function is identically 1, we just need to find the integral

$$\iint\limits_C z\gamma \; dS = \iint\limits_C z \; dS$$

and divide it by the total mass of the cone,

$$\iint\limits_C \gamma \; dS = \iint\limits_C 1 \; dS = \pi\sqrt{2}$$

(from our above calculations.)

Using the definition of the integral, and our earlier calculation that $\left|\left|\frac{\partial\varphi}{\partial z} \times \frac{\partial\varphi}{\partial\theta}\right|\right| = z\sqrt{2}$, we have that

$$\iint_C z \, dS = \int_0^1 \int_0^{2\pi} z \cdot \left|\left|\frac{\partial \varphi}{\partial z} \times \frac{\partial \varphi}{\partial \theta}\right|\right| d\theta dz$$
$$= \int_0^1 \int_0^{2\pi} z^2 \sqrt{2} d\theta dz$$
$$= \int_0^1 2\pi z^2 \sqrt{2} dz$$
$$= 2\pi\sqrt{2}/3.$$

So, the z-coördinate of our center of mass is just

$$\frac{2\pi\sqrt{2}/3}{\pi\sqrt{2}} = 2/3.$$

So, if you have a right cone with uniform density (say, you made your cone out of paper), and you puncture it about 2/3-rds of the way up with a pencil or dowel or somesuch thing, it should spin freely about that axis, as its weight is equally distributed on all sides.

2.4 Stokes' theorem.

Stokes' theorem, roughly speaking, is Green's theorem for surfaces. Explicitly, it's the following statement:

Theorem 1 *(Stokes' theorem) Suppose that S is a bounded surface*[2] *with boundary given by the counterclockwise-oriented curve C, and $F : \mathbb{R}^3 \to \mathbb{R}^3$ is some continuously differentiable function. Then, we have the following equality:*

$$\iint_S ((\nabla \times F) \cdot n) \, dS = \int_C F \cdot ds,$$

where (as always) n denotes the unit normal vector at any point on S. (It bears noting that if we have a parametrization φ of our surface S, we can explicitly write this vector n as $\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} / \left|\left|\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y}\right|\right|$.)

In practice, we use Stokes' theorem in pretty much all of the same cases that we use Green's theorem:

- Turning integrals of functions over really awful curves into integrals of curls of functions over surfaces. Often, this process of taking a curl will make our function 0 or at the least quite trivial.

- Turning integrals of really awful functions over some curve into integrals of curls (which will sometimes become much simpler, because you're taking derivatives) over some region.

[2] A set S is called **bounded** if there is some n such that $||s|| < n$, for all $s \in S$

- If you're integrating something of the form $(\nabla \times f) \cdot n$ over a surface, you can of course go backwards, which will often make life easier (as finding the curl of f and the unit normal vector will usually be a fairly obnoxious process.) In practice, this might not come up too often, as it's not always obvious when a given expression is a curl, or the dot product of a curl with a normal vector; so don't look for this unless you're really stuck, or the problem explictly gives you your function in the form $(\nabla \times f) \cdot n$.

To illustrate how this goes, we work an example:

Example. If $F(x, y, z) = (-xy^2, x^2y, z)$ and S is the sphere cap $\{(x, y, z) : x^2 + y^2 + z^2 = 25, z \geq 4\}$, find the integral of $(\nabla \times F) \cdot n$ over S.

Solution. As you have probably noticed in the past few HW's, sphere caps are kind of pernicious things. So, instead of integrating over this one, let's use Stokes' theorem to instead integrate along its boundary!

Specifically: the sphere cap above has boundary

$$\partial S = \{(x, y, z) : x^2 + y^2 + z^2 = 25, z = 4\} = \{(x, y, z) : x^2 + y^2 = 3^2, z = 4\},$$

which is traversed in the counterclockwise direction by the curve $\gamma(\theta) = (3\cos(\theta), 3\sin(\theta), 4)$. So, we can use Stokes' theorem to say that

$$\begin{aligned}
\iint_S (\nabla \times F) \cdot n dS &= \int_C F dc \\
&= \int_0^{2\pi} (-xy^2, x^2y, z)\Big|_{\gamma(\theta)} \cdot \gamma'(\theta) d\theta \\
&= \int_0^{2\pi} (-27\cos(\theta)\sin^2(\theta), 27\cos^2(\theta)\sin(\theta), 4) \cdot (-3\sin(\theta), 3\cos(\theta), 0) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin^3(\theta) + 81\cos^3(\theta)\sin(\theta) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta)(\sin^2(\theta) + \cos^2(\theta)) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta) d\theta \\
&= \int_0^{2\pi} \frac{81\sin(2\theta)}{2} d\theta \\
&= 0.
\end{aligned}$$