Math 1c	TA: Padraic Bartlett

Recitation 10: Final Review - Definitions

Week 10 *Caltech 2012*

The Ma1c final is focused on testing the material presented in the second half of this course: in other words, it's a ton of questions about integrals! This handout is an attempt to summarize everything we've discussed about the integral in the second half of this quarter.

1. **Types of integrals.** We've learned how to take several kinds of integrals in this course:

   - **"Normal" integrals.** Given a region $R \subset \mathbb{R}^n$, we know how to take the integral of any function $F : \mathbb{R}^n \to \mathbb{R}^m$ over such a region by taking **iterated integrals**. For example, if R is some sort of a n-dimensional box $[a_1, b_1] \times \ldots [a_n, b_n]$, we can write $\iint_R F dV$ as the iterated integral

     $$\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} F dx_n \ldots dx_1.$$

     Part of being able to do these integrals is the ability to describe a region R via sets of nested parameters. For example, if R is the upper-right quadrant of the unit disk

     $$R = \{(x, y) : x^2 + y^2 \leq 1, 0 \leq x, 0 \leq y\},$$

     you should be able to describe R as the set of all points such that

     $$x \in [0, 1], y \in [0, \sqrt{1 - x^2}],$$

     and therefore notice that that we can express

     $$\iint_R f(x, y) dy dx = \int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) dy dx,$$

     for some function f. Be able to do this "nested parameter" thing over most kinds of regions: usually, the way you do this is by picking one variable, determining its maximum range, then (for some fixed value of that first variable) pick a second variable and determine its maximum range depending on the first variable, and so on/so forth.

   - **Line integrals.** Given a parametrized curve $\gamma : [a, b] \to \mathbb{R}^n$, we can find the integral of either a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ or a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ along this curve. Specifically, we can express these integrals as the following:

     $$\int_\gamma F \cdot d\gamma = \int_a^b (F \circ \gamma(t)) \cdot (\gamma'(t)) dt, \quad \text{and}$$

     $$\int_\gamma f \; d\gamma = \int_a^b (f \circ \gamma(t)) ||\gamma'(t)|| dt.$$

- **Surface integrals.** Given a parametrized surface S with parametrization $T : R \to S$, $R \subseteq \mathbb{R}^2$, we can find the integral of any function $f : \mathbb{R}^n \to \mathbb{R}$ over S, as well as the integral of any vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ over S. Specifically, we can express the integral of f over S as the following two-dimensional integral over R:

$$\iint_S f dS = \iint_R (f \circ T(u,v)) \cdot ||T_u \times T_v||\, dudv.$$

As well, recall that a unit normal vector to our surface, $\ltimes$, can be given by the formula

$$\mathbf{n} = \frac{(T_u \times T_v)}{||T_u \times T_v||} \text{ or } \frac{(T_v \times T_u)}{||T_v \times T_u||}$$

up to the **orientation** of $\mathbf{n}$: i.e. depending on whether we look at $(T_u \times T_v)$ or $(T_v \times T_u)$, we will get either $\mathbf{n}$ or $-\mathbf{n}$. Choosing an **orientation** for our surface S is simply choosing which of these two choices of normal vectors we will make for our entire integral: whenever we ask you to integrate a vector field over a surface, we will tell you what orientation you should pick (i.e. by asking you to orient S so that "the normals point away from the origin," or something like that.) Once you've fixed an orientation, say the $T_u \times T_v$ one, we define the integral of F over S as the following integral:

$$\begin{aligned}\iint_S F \cdot dS = \iint_S F \cdot \mathbf{n} dS &= \iint_R (f \circ T(u,v)) \cdot \frac{(T_u \times T_v)}{||T_u \times T_v||} \cdot ||T_u \times T_v||\, dudv \\ &= \iint_R (f \circ T(u,v)) \cdot (T_u \times T_v)\, dudv.\end{aligned}$$

The trickiest thing going on here is "how" you **choose** your parametrization. For finding a parametrization of a surface S, you can usually do one of the following two things:

  - Often, if you describe your surface S in cylindrical or spherical coördinates, you'll see that one of the coördinates you're describing your surface in is constant. For example, a spherical shell of radius 3 can be described in spherical coördinates as the set of all point $(3, \theta, \phi)$, where $\theta \in [0, 2\pi], \phi \in [0, \pi]$. In this kind of situation, our parametrization is just using this coördinate system with the constant variable treated as a constant: i.e. for the spherical shell of radius 3, our parametrization is just

$$T(\theta, \phi) = (3\cos(\theta)\sin(\phi), 3\sin(\theta)\sin(\phi), 3\cos(\phi)),$$

    where $\theta \in [0, 2\pi], \phi \in [0, \pi]$.

- If this doesn't work out, the other tactic that's often useful is finding an equation that describes your surface, and solving for one of the variables in terms of the others. For example, suppose that we're looking at the surface S given by the upper sheet of the hyperboloid of two sheets between heights 1 and 2: i.e.

$$S = \{(x, y, z) : -x^2 - y^2 + z^2 = 1, z \in [1, 2]\}.$$

  In this case, because z is positive, we can solve for z in terms of the other variables, and express S as

$$S = \{(x, y, z) : z = \sqrt{1 + x^2 + y^2}, z \in [1, 2]\}.$$

  We can then use this to formulate a parametrization of S: simply let x and y range over the possible values that keep z between 1 and 2, and then set $z = \sqrt{1 + x^2 + y^2}$:

$$T(x, y) = (x, y, \sqrt{1 + x^2 + y^2}), x \in [-\sqrt{3}, \sqrt{3}], y \in [-\sqrt{3 - x^2}, \sqrt{3 - x^2}].$$

  You can of course combine these two approaches: for example, if we were to use cylindrical coördinates on our surface S above and replace x with $r\cos(\theta), y$ with $r\sin(\theta)$, we can see that we can easily express T instead as the map

$$T(r, \theta) = (r\cos(\theta), r\sin(\theta), \sqrt{1 + r^2}), r \in [0, \sqrt{3}], \theta \in [0, 2\pi],$$

  which may be easier to work with.

2. **Tools for evaluating integrals.** Throughout Ma1c, you've ran into many integrals of the above kinds that were difficult or impossible to directly evaluate. Motivated by these problems, we developed a number of theorems and tools about integration, which we repeat here:

   - **Green's theorem.** There are a number of forms of Green's theorem; we state the simpler and most commonly used version here. Suppose that R is a region in $\mathbb{R}^2$ with boundary ∂R given by the simple closed curve C, and suppose that γ is a traversal of C in the counterclockwise direction. Suppose as well that P and Q are a pair of C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}$. Then, we have the following equality:

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\gamma (Pdx + Qdy).$$

   - **Stokes' theorem.** Stokes' theorem, quite literally, is Green's theorem for surfaces in $\mathbb{R}^3$ (as opposed to restricting them to lying in the plane $\mathbb{R}^2$.) Specifically, it is the following claim: suppose that S is a surface in $\mathbb{R}^3$ with boundary ∂S given by the simple closed curve C, suppose that $\mathbf{n}$ is a unit normal vector to S that gives S some sort of orientation, and suppose that γ is a traversal of C such that the interior of S always lies on the left of γ's forward direction, assuming that we're viewing the surface such that the normal vector $\mathbf{n}$ is pointing towards us. Suppose as well that F is a vector field from $\mathbb{R}^3$ to $\mathbb{R}^3$. Then, we have the following equality:

$$\iint_S (\nabla \times F) \cdot \mathbf{n}\ dS = \int_\gamma F d\gamma.$$

In general, you use Green's and Stokes's theorems whenever you have a integral of a function over an awful curve (and taking derivatives to work with your function over a region, which is what the curl does, will make things easier), or you have an integral of a curl-like function over an awful region (and working on the curve would make things easier.)

- **Divergence/Gauss's theorem.** Let W be a region in $\mathbb{R}^3$ with boundary given by some surface S, let $\mathbf{n}$ be the outward-pointing (i.e. away from W) unit normal vector to S, and let F be a smooth vector field defined on W. Then

$$\iiint_W (\text{div}(F)) dV = \iint_{\partial W} (F \cdot \mathbf{n}) dS.$$

Again, use this like you would use Green's and Stokes's theorems.

- **Change of variables.** A common tactic to make integrals easier is to apply the technique of change of variables, which allows us to describe regions in $\mathbb{R}^n$ using coördinate systems other than the standard Euclidean ones. In general, the change-of-variables theorem says the following:
  - Suppose that R is an open region in $\mathbb{R}^n$, g is a C^1 map $\mathbb{R}^n \to \mathbb{R}^n$ on an open neighborhood of R, and that f is a continuous function on an open neighborhood of the region $g(R)$. Then, we have

$$\int_{g(R)} f(\mathbf{x}) dV = \int_R f(g(\mathbf{x})) \cdot \det(D(g(\mathbf{x}))) dV.$$

Specifically, the three most common change-of-variable choices are transitions to the **polar**, **cylindrical**, and **spherical** coördinate systems, which we review here:

  - **Polar coördinates.** Suppose that R is a region in $\mathbb{R}^2$ described in polar coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi)$ such that $\gamma(A) = R$, where γ is the polar coördinates map $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$. Then, for any integrable function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\iiint_{\gamma(A)} f(x, y) dV = \iiint_A f(r\cos(\theta), r\sin(\theta)) \cdot r\ dV.$$

  - **Cylindrical coördinates.** Suppose that R is a region in $\mathbb{R}^3$ described in cylindrical coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi) \times (-\infty, \infty)$

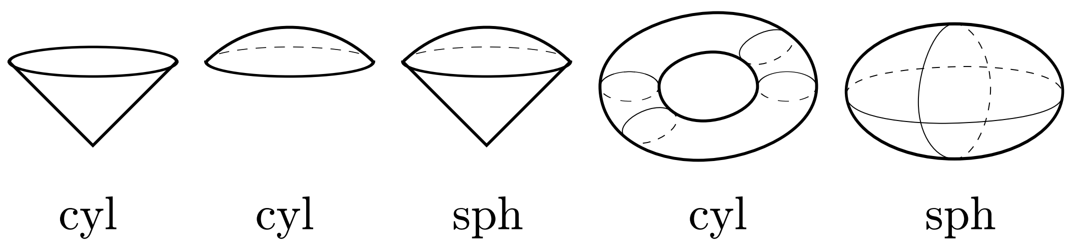
such that $\gamma(A) = R$, where γ is the cylindrical coördinates map $(r, \theta, z) \mapsto (r\cos(\theta), r\sin(\theta), z)$. Then, for any integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint\limits_{\gamma(A)} f(x, y) dV = \iiint\limits_{A} f(r\cos(\theta), r\sin(\theta), z) \cdot r \; dV.$$

– **Spherical coördinates.** Suppose that R is a region in $\mathbb{R}^3$ described in spherical coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi) \times [0, \pi)$ such that $\gamma(A) = R$, where γ is the spherical coördinates map $(r, \theta, \varphi) \mapsto (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$. Then, for any integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint\limits_{\gamma(A)} f(x, y) dV = \iiint\limits_{A} f(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)) \cdot r^2 \sin(\varphi) \; dV.$$

One of the trickiest things to do with change of variables is deciding which coördinate system to use on a given set. For example, consider the following five shapes:



To describe the cone, sphere cap, or torus above, cylindrical coördinates are probably going to lead to the easiest calculations. Why is this? Well, all three of these shapes have a large degree of symmetry around their z-axis; therefore, we'd expect it to be relatively easy to describe these shapes as a collection of points (r, θ, z). However, these shapes do *not* have a large degree of rotational symmetry: in other words, if we were to attempt to describe them with the coördinate (r, θ, φ), we really wouldn't know where to begin with the φ coördinate.

However, for the ellipsoid and "ice-cream-cone" section of the ellipsoid, spherical coördinates are much more natural: in these cases, it's fairly easy to describe these sets as collections of points of the form (r, θ, φ).

In general, if you're uncertain which of the two to try, simply pick one and see how the integral goes! If you chose wisely, it should work out; otherwise, you can always just go back and try the other coördinate system.

3. **Applications of the integral.** Finally, it bears noting that we've developed a few applications of the integral to finding volume, surface area, length, and centers of mass. We review these here:

   • **Volume, surface area, and length**. If you have a solid V, a surface S, or a curve C, you can find the volume/area/length of your object by integrating the function 1 over that object.

- **Area, via Green's theorem.** If you have a region $R \subset \mathbb{R}^2$ with boundary given by the counterclockwise-oriented curve γ, you can use Green's theorem to find its area as a line integral. Specifically, notice that if $F(x,y) = \left(-\frac{y}{2}, \frac{x}{2}\right)$, we have $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and therefore that Green's theorem says that

$$\iint\limits_R 1 \; dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma.$$

- **Center of mass.** Suppose that an object A (a curve, surface, or solid) has density function $\delta(\mathbf{x})$. Then, the x_i-coördinate of its center of mass is given by the ratio

$$\frac{\int\limits_A x_i \delta(\mathbf{x}) \; dA}{\int\limits_A \delta(\mathbf{x}) \; dA}.$$

This is pretty much everything we've covered in the second half of our course with respect to the integral! The only other topic we've discussed since the midterm are the operations of **div** and **curl**, which we quickly review here:

1. **Div and curl: definitions.** Given a C^1 vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, we can defind the **divergence** and **curl** of F as follows:

   - **Divergence**. The **divergence** of F, often denoted either as $\text{div}(F)$ or $\nabla \cdot F$, is the following function $\mathbb{R}^3 \to \mathbb{R}$:

   $$\text{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

   - **Curl.** The **curl** of F, denoted $\text{curl}(F)$ or $\nabla \times F$, is the following map $\mathbb{R}^3 \to \mathbb{R}^3$:

   $$\text{curl}(F) = \nabla \times F = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right).$$

   Often, the curl is written as the "determinant" of the following matrix:

   $$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

2. **Theorems**. We have a pair of rather useful theorems about the divergence and curl of functions, which we state here:

   - For any C^2 function F, $\text{div}(\text{curl}(F))$ is always 0.

- For any C^2 function F, $\text{curl}(\text{grad}(F))$ is always 0.

These theorems are a pair of very useful tests that can often tell us that a given function F is not a conservative vector field (i.e. a gradient) or a curl of some other function. For example, if we examined the function $F(x, y, z) = (x, y, z)$, we can immediately tell that F is not the curl of some other function because its divergence is $1 + 1 + 1 \neq 0$.