Math 1c	TA: Padraic Bartlett

Recitation 7: Change of Variables; Path and Line Integrals

Week 7 — *Caltech 2012*

1 Random Question

1. Show that the 5-dimensional unit ball $B_5 = \{\mathbf{x} \in \mathbb{R}^5 : ||x|| \leq 1\}$ has volume $8\pi^2/15$.

2. Show that this volume is the largest volume attained by the n-dimensional unit spheres: i.e. show that for any $B_n = \{\mathbf{x} \in \mathbb{R}^n : ||x|| \leq 1\}$, vol$(B_n)$ <vol(B_5).

2 Change of Variables

2.1 Change of variables: the theorem and motivation.

(Here, we discuss a little bit of "why" our change of variables formula is what it is. Feel free to skip to the statement of the theorem at the end of this subsection, if you'd rather hurry up and get to the examples!)

The concept of "changing variables" is one we've ran into in single-variable calculus:

Theorem 1 *(Change of variables, single-variable form) Suppose that f is a continuous function over the interval $(g(a), g(b))$, and that g is a $1-1$ continuous map with continuous derivative from (a, b) to $(g(a), g(b))$. Then, we have that*

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x)) \cdot g'(x)dx.$$

The idea here, roughly, was the following: the integral of f over the interval $(g(a), g(b))$ is the same as the integral of $f \circ g$ over the interval (a, b), as long as we correct for how g "distorts space." In other words, on the left (where g's been applied to the domain (a, b)), we're integrating with respect to dx, the change in x: however, on the right, we're now integrating $f \circ g$, and therefore we should integrate with respect to $d(g(x)) = g'(x)dx$.

So: in multiple variables, we want to have a similar theorem! Basically, given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$, a domain $R \subset \mathbb{R}^n$, and a differentiable map $g : \mathbb{R}^n \to \mathbb{R}^n$, we want a way to relate the integral of f over $g(R)$ and the integral of $f \circ g$ over R.

How can we do this? In other words, how can we correct for how g "distorts space," like we did for our single-variable case? Well: locally, we know that small changes in the vector $\mathbf{x}$ are measured by $D(g(\mathbf{x}))$, the $n \times n$ matrix of partial derivatives of g. Specifically, from Math 1b, we know that $\det(D(g(\mathbf{x})))$ measures the volume of the image of the unit cube under the map $D(g(\mathbf{x}))$. So, in a sense, this quantity – $\det(D(g(\mathbf{x})))$, the determinant of the Jacobian of g – is telling us how much g is locally inflating or shrinking space at the point $\mathbf{x}$! So, we might hope that this is the correct quantity to scale by. As it turns out, it is! Specifically, we have the following theorem:

Theorem 2 *(Change of variables, multiple-variable form) Suppose that R is a region in $\mathbb{R}^n$, g is a $1-1$ map with continuous partial derivatives that maps R to some region $g(R) \subset \mathbb{R}^n$, and that f is a continuous function. Then, we have*

$$\int_{g(R)} f(\mathbf{x})dV = \int_R f(g(\mathbf{x})) \cdot \det(Dg(\mathbf{x}))dV.$$

This, as you may have noticed, is not precisely the theorem in your textbook. That theorem reads as follows:

Theorem 3 *(Change of variables, multiple-variable form) Suppose that R is a region in $\mathbb{R}^n$, g is a 1-1 map with continuous partial derivatives that maps $g^{-1}(R)$ to R, and that f is a continuous function. Then, we have*

$$\int_R f(\mathbf{x})dV = \int_{g^{-1}(R)} f(g(\mathbf{x})) \cdot \det(Dg(\mathbf{x}))dV.$$

The change between the two theorems lies in how you're thinking of the integral you've started with: you can consider it either as a integral where you've already got a good guess for what your g-map will be ($\int_{g(R)} f(\mathbf{x})dV$), or you can think of it in the situation where you don't have an idea what your g-map is yet ($\int_R f(\mathbf{x})dV$).

Also! I didn't emphasize this enough in recitation: if you use either of these results, you need to be very very careful to insure that your map g is 1-1 on your region R! Otherwise, it's possible that g^{-1} will "fold" parts of R on top of each other, in such a way that the integral at the right will no longer be over something that looks like R. For example, if your map g was the map $(x, y) \mapsto (x^2, y^2)$, and you were to try applying this map to the integral

$$\int_{g([-1,1]\times[0,1])} 1 \, dV,$$

you'd get

$$\int_{g([-1,1]\times[0,1])} 1 \, dV =^{??} \int_{[-1,1]\times[0,1]} 1 \cdot \det \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} dV = \int_{-1}^{1} \int_0^1 (4xy) dydx = 2,$$

even though

$$\int_{g([-1,1]\times[0,1])} 1 \, dV = \int_{[0,1]^2} 1 \, dV = \int_0^1 \int_0^1 dxdy = 1.$$

This is because the map g "folds" the region $[-1, 1] \times [0, 1]$ we were integrating over in half: therefore, if we use change of variables to change the region we're integrating over from $g([-1, 1] \times [0, 1])$ to $[-1, 1] \times [0, 1]$, we'd expect to see an "unfolding", which would cause our integral to double. (Which is precisely what we saw!)

2.2 Common variable changes.

There are three exceptionally common changes of variable, which we review here briefly:

Theorem 4 *(Change of variables, polar:) Let $\gamma : [0, \infty) \times [0, 2\pi)$ be the polar coördinates map $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$. Then $D(\gamma(r, \theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$, $\det(D(\gamma(r, \theta))) = r$, and we have*

$$\int_{\gamma(R)} f(x, y) dV = \int_R f(r\cos(\theta), r\sin(\theta)) \cdot r dV,$$

for any region $R \subset [0, \infty) \times [0, 2\pi)$, and any continuous function f on an open neigborhood of R.

In other words, if we have a region R described by polar coördinates, we can say that the integral of f over $\gamma(R)$ is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over this region interpreted in Euclidean coördinates. For example, suppose that R was the unit disk, which we can express using our polar coördinates map as $\gamma([0, 1] \times [0, 2\pi))$. Then, change of variables tells us that the integral of f over the unit disk is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over the Euclidean-coördinates rectangle $[0, 1] \times [0, 2\pi)$.

Cylindrical coördinates are similar:

Theorem 5 *(Change of variables, cylindrical:) Let $\gamma : [0, \infty) \times [0, 2\pi) \times \mathbb{R}$ be the cylindrical coördinates map $(r, \theta, z) \mapsto (r\cos(\theta), r\sin(\theta), z)$. Then $D(\gamma(r, \theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(D(\gamma(r, \theta))) = r$, and we have*

$$\int_{\gamma(R)} f(x, y) dV = \int_R f(r\cos(\theta), r\sin(\theta), z) \cdot r dV,$$

for any region $R \subset [0, \infty) \times [0, 2\pi) \times (-\infty, \infty)$, and any continuous function f on an open neigborhood of R.

Spherical coördinates are a bit trickier, but have a similar form:

Theorem 6 *(Change of variables, spherical:) Let $\gamma : [0, \infty) \times [0, \pi) \times [0, 2\pi)$ be the cylindrical coördinates map $(r, \varphi, \theta) \mapsto (r\cos(\varphi), r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta))$. Then $D(\gamma(r, \theta)) = \begin{bmatrix} \cos(\varphi) & -r\sin(\varphi) & 0 \\ \sin(\varphi)\cos(\theta) & r\cos(\varphi)\cos(\theta) & -r\sin(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) & r\cos(\varphi)\sin(\theta) & r\sin(\varphi)\cos(\theta) \end{bmatrix}$, $\det(D(\gamma(r, \theta))) = r^2\sin(\varphi)$, and we have*

$$\int_{\gamma(R)} f(x, y) dV = \int_R f(r\cos(\varphi), r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta)) \cdot r^2\sin(\varphi) dV,$$

for any region $R \subset [0, \infty) \times [0, \pi) \times [0, 2\pi)$ and any continuous function f on an open neigborhood of R.

There are a few other coördinate transforms that will often come up:

- Various translations of space: i.e. maps $(x, y, z) \mapsto (x + c_1, y + c_2, z + c_3)$. The determinant of the Jacobian of such maps will always be 1.
- Various ways to scale space: i.e. maps $(x, y) \mapsto (\lambda_1 x, \lambda_2 y)$. The determinant of the Jacobian of such maps will be the product of these scaling constants $\lambda_1 \cdot \ldots \cdot \lambda_n$.
- Various compositions of these maps: i.e. a translation map, followed by a spherical coördinates map, followed by a scaling map. Using the chain rule, the determinant of the Jacobian of any such composition of maps is just the product of the determinants of the individual Jacobians.

Often, the trickiest thing to do in a problem is realize which coördinate system makes the most sense to use, and then to use it. We work two examples below:

2.3 Examples.

Example. Choose a random point in the upper-right quadrant of the unit disk: i.e. a random point (x, y) such that $x, y \geq 0, x^2 + y^2 \leq 1$. What is the average value of the minimum of (x, y)?

Solution. We are looking for the average value of the function $\min(x, y)$ over this upper-right quadrant of the unit disk, i.e.

$$\frac{1}{\text{area(part of unit disk)}} \int_{\text{(part of unit disk)}} \min(x, y) dA.$$

Polar coördinates look like a good candidate for how we can find this integral! In particular, if we apply the change of variables formula using the polar coördinate transform $g(r, \theta) = (r \cos(\theta), r \sin(\theta))$ to the above integral, we get

$$\begin{aligned} \int_{\text{(part of unit disk)}} \min(x, y) dA &= \int_0^1 \int_0^{\pi/2} \min(r \cos(\theta), r \sin(\theta)) \cdot \det(Dg) \cdot d\theta dr \\ &= \int_0^1 \int_0^{\pi/2} \min(r \cos(\theta), r \sin(\theta)) \cdot r \cdot d\theta dr. \end{aligned}$$

Note that we can do because the polar-coördinate transform is $1-1$ on the set $[0, 1] \times [0, \pi/2]$, which is precisely the set that maps onto the upper-right quadrant of the unit disk.

With this done, we can simply integrate. First, notice that from 0 to $\pi/4$, $\sin(\theta) \leq \cos(\theta)$, and from $\pi/4$ to π, we have that $\cos(\theta) \leq \sin(\theta)$. Therefore, if we break our inner integral into two parts, one going from 0 to $\pi/4$ and the other from $\pi/4$ to $\pi/2$, we have

$$\begin{aligned}
&\int_0^1 \int_0^{\pi/2} r\min(r\cos(\theta), r\sin(\theta))\ d\theta dr\\
&= \int_0^1 \left(\int_0^{\pi/4} r\min(r\cos(\theta), r\sin(\theta)) d\theta + \int_{\pi/4}^{\pi/2} r\min(r\cos(\theta), r\sin(\theta)) d\theta \right) dr\\
&= \int_0^1 \left(\int_0^{\pi/4} r^2\sin(\theta)d\theta + \int_{\pi/4}^{\pi/2} r^2\cos(\theta)d\theta \right) dr\\
&= \int_0^1 \left(r^2(-\cos(\theta))\Big|_0^{\pi/4} + r^2(\sin(\theta))\Big|_{\pi/4}^{\pi/2} \right) dr\\
&= \int_0^1 r^2\left(1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1\right) dr\\
&= (2-\sqrt{2})\frac{r^3}{3}\Big|_0^1\\
&= \frac{2-\sqrt{2}}{3}.
\end{aligned}$$

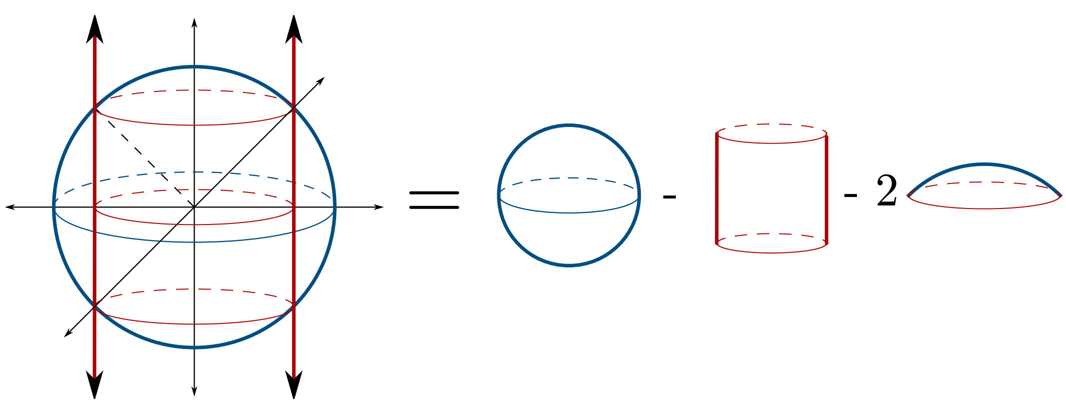
Therefore, if we want the average value of $\min(x, y)$ over this upper-right quadrant, we just have to divide our integral by the area of the upper-right quadrant, $\pi/4$. In other words, we've just proven that the average value of $\min(x, y)$ is

$$\frac{8-4\sqrt{2}}{3\pi},$$

which is roughly $\frac{1}{4}$.

Example. Let $B_3 = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$ be the unit ball and $C = \{(x, y, z) : x^2 + y^2 \le 1/4\}$ be a cylinder of radius $1/2$ around the z-axis. Find the volume of the set of points $R = B_3 \setminus C$: i.e. the collection R of points both contained within the unit ball B_3 and outside of the cylinder C.

Solution. So: to set this problem up, let's first draw a picture of the situation:



We want to find the volume trapped between the cylinder and the sphere. To do this, we can simply take the volume of the unit sphere and subtract off the collection of points trapped within both the sphere in the cylinder. We can decompose those points into two kinds:

- The points trapped within the red "can" $\{(x, y, z) : x^2 + y^2 \leq 1/4, -\sqrt{3}/2 \leq z \leq \sqrt{3}/2\}$.
- The points trapped within one of the two red and blue "sphere caps" $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1, z > \sqrt{3}/2\}$ or $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1, z < -\sqrt{3}/2\}$.

To find the volume of R, then, it suffices to find these three volumes, and then subtract them off of each other.

For fun, we start by rederiving the formula for the area of a sphere, using spherical coördinates:

$$\begin{aligned}
\int_{B_3} 1dV &= \int_{[0,1]\times[0,\pi)\times[0,2\pi)} 1 \cdot r^2 \sin(\varphi) dV \\
&= \int_0^1 \int_0^\pi \int_0^{2\pi} r^2 \sin(\varphi) d\theta d\varphi dr \\
&= \int_0^1 \int_0^\pi 2\pi r^2 \sin(\varphi) d\varphi dr \\
&= \int_0^1 \left(2\pi r^2 \cos(\varphi)\right)\Big|_0^\pi dr \\
&= \int_0^1 4\pi r^2 dr \\
&= \frac{4\pi}{3}.
\end{aligned}$$

Now, we use cylindrical coördinates to find the volume of our "can":

$$\begin{aligned}
\int_{\text{“can”}} 1dV &= \int_{[0,1/2]\times[0,2\pi)\times[-\sqrt{3}/2,\sqrt{3}/2]} 1 \cdot r\ dV \\
&= \int_0^{1/2} \int_0^{2\pi} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} r\ dz d\theta dr \\
&= \int_0^{1/2} \int_0^{2\pi} r\sqrt{3}\ d\theta dr \\
&= \int_0^{1/2} 2r\pi\sqrt{3} dr \\
&= \frac{\pi\sqrt{3}}{4}.
\end{aligned}$$

Finally, we need to find the volume of our sphere caps. You could attempt to use spherical coördinates to describe these sphere caps, but the equations for what constraints we would have on r will involve some somewhat painful trig equations. However, if we use cylindrical coördinates, we can see that the northern sphere cap can be described as the set of points $\{(r,\theta,z) : 0 \le r \le 1/2, \sqrt{3}/2 \le z \le \sqrt{1-r^2}\}$. This is because the points in our sphere cap are those with (x,y) part inside of our cylinder, and z- coördinate between the floor $z = \sqrt{3}/2$ of the sphere cap and the ceiling $z^2 + r^2 = 1$ of the sphere cap.

So, this is not such a bad set! Specifically, if we use this description of the sphere cap and apply change of variables, we have

$$
\begin{aligned}
\int_{\text{“sphere cap”}} 1 dV &= \int_0^{1/2} \int_0^{2\pi} \int_{\sqrt{3}/2}^{\sqrt{1-r^2}} r \; dz d\theta dr \\
&= \int_0^{1/2} \int_0^{2\pi} \left(\sqrt{1-r^2} - \sqrt{3}/2\right) r \; d\theta dr \\
&= \int_0^{1/2} 2\pi \left(\sqrt{1-r^2} - \sqrt{3}/2\right) r \; dr \\
&= \left(\int_0^{1/2} 2\pi r\sqrt{1-r^2} \; dr\right) - \left(\int_0^{1/2} \pi r\sqrt{3} dr\right) \\
&= \left(\int_0^{1/2} 2\pi r\sqrt{1-r^2} \; dr\right) - \frac{\pi\sqrt{3}}{8} \\
&= \left(\int_1^{3/4} -\pi\sqrt{u} \; du\right) - \frac{\pi\sqrt{3}}{8} \\
&= \left(\int_{3/4}^{1} \pi\sqrt{u} \; du\right) - \frac{\pi\sqrt{3}}{8} \\
&= \left(\frac{2\pi}{3} u^{3/2}\right)\Bigg|_{3/4}^{1} - \frac{\pi\sqrt{3}}{8} \\
&= \frac{2\pi}{3} - \frac{\pi\sqrt{3}}{4} - \frac{\pi\sqrt{3}}{8} \\
&= \frac{2\pi}{3} - \frac{3\pi\sqrt{3}}{8}
\end{aligned}
$$

where we used the substitution $u = 1-r^2, du = -2r$ to evaluate the first of our two integrals.

So, in summary, the volume of our set R is

$$
\begin{aligned}
&\frac{4\pi}{3} - \frac{\pi\sqrt{3}}{4} - 2\left(\frac{2\pi}{3} - \frac{3\pi\sqrt{3}}{8}\right) \\
=&\frac{\pi\sqrt{3}}{8}.
\end{aligned}
$$

3 Line and Path Integrals

With the tools above, we have an idea about how we can integrate functions over lots of different kinds of regions! Basically, the technique of change-of-variables gives us a way to integrate functions over any region that we can describe in some clever way as $g(R)$. However, in the section above, we restricted ourselves to looking at maps $g : \mathbb{R}^n \to \mathbb{R}^n$. What if we look at other kinds of maps? In particular, what if we let $g = \gamma : [a, b] \to \mathbb{R}^n$ be a **curve**: could we come up with some clever way to integrate functions over this kind of region?

As it turns out, we can! Consider the following definition:

Definition. Suppose that we have a function $f : \mathbb{R}^n \to \mathbb{R}$ and a continuous path $\gamma : [a, b] \to \mathbb{R}^n$. Then, we can define the **path integral** of f along γ as

$$\int_\gamma f \cdot d\gamma := \int_a^b f(\gamma(t)) \cdot ||\gamma'(t)||dt.$$

This integral gives you the "average" value of f along the curve γ, scaled by the length of γ itself.

The above definition arises from the same set of ideas we used in our change-of-variables discussion: to integrate a function f over some $\gamma([a, b])$, it suffices to integrate $f \circ \gamma$ over $[a, b]$, as long as we scale for how γ "distorts space:" i.e. we multiply by $||\gamma'(t)||$.

Using similar notions, we can also come up with the following definition of a **line integral**, which gives us a way to integrate a **vector field** $F : \mathbb{R}^n \to \mathbb{R}^n$ over a given curve:

Definition. For a function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a continuous path $\gamma : [a, b] \to \mathbb{R}^n$, we define the **line integral** of F along γ as

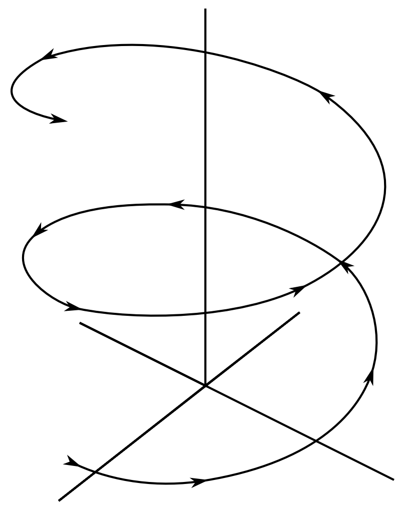
$$\int_\gamma F \cdot d\gamma := \int_a^b F(\gamma(t)) \cdot \gamma'(t)dt$$

You can interpret this integral as the "work" done by F along the curve γ. This is because we're integrating the dot product of F and γ', and the dot product of a "force" vector (F) and a "direction" vector (γ') is just the work done by that force in that direction.

We illustrate the uses of these two definitions with a pair of examples:

Example. Integrate the function $f(x, y, z) = x^2y^2 + y^2z^2 + x^2z^2$ over the helix $\gamma(t) = (\cos(t), \sin(t), t), t \in [0, 2\pi)$.

Solution. Because it's pretty, we sketch this curve here:

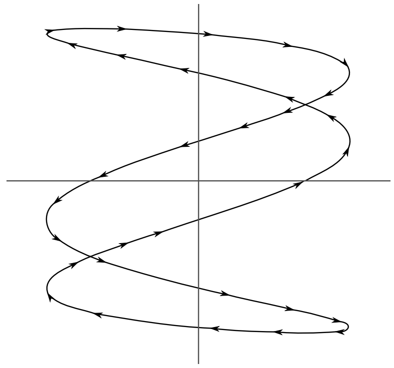


With that done, we can just simply apply our above definition for the path integral:

$$
\begin{aligned}
\int_\gamma f(x,y,z)\cdot d\gamma &= \int_0^{2\pi} f(\cos(t),\sin(t),t)\cdot ||(\cos(t),\sin(t),t)'||dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t) + t^2\sin^2(t) + t^2\cos^2(t))\cdot ||(-\sin(t),\cos(t),1)||dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t) + t^2)\cdot \sqrt{\sin^2(t)+\cos^2(t)+1^2}dt \\
&= \int_0^{2\pi} \left(\frac{\sin^2(2t)}{4} + t^2\right)\cdot \sqrt{2}dt \\
&= \int_0^{2\pi} \left(\frac{1-cos(4t)}{8} + t^2\right)\cdot \sqrt{2}dt \\
&= \left(\frac{t}{8} - \frac{\sin(4t)}{32} + \frac{t^3}{3}\right)\sqrt{2}\Bigg|_0^{2\pi} \\
&= \left(\frac{2\pi}{8} - \frac{0}{32} + \frac{(2\pi)^3}{3}\right)\sqrt{2} - 0 \\
&= \frac{2\pi\sqrt{2}}{8} + \frac{8\pi^3\sqrt{2}}{3}.
\end{aligned}
$$

Example. Integrate the vector field $F(x,y) = (x^2,y^2)$ over the Lissajous curve $\gamma(t) = (\sin(3t+\pi/4),\sin(t))$, where $t \in [0,2\pi)$.

Solution. Because it's also pretty, we sketch this curve as well:



Now, because we're integrating a vector field, we use the line integral:

$$\begin{aligned}
\int_\gamma F(x,y)dc &= \int_0^{2\pi} (F \circ \gamma(t)) \cdot \gamma'(t)dt \\
&= \int_0^{2\pi} \left(\sin^2(3t+\pi/4), \sin^2(t)\right) \cdot (3\cos(3t+\pi/4), \cos(t))\,dt \\
&= \int_0^{2\pi} 3\cos(3t+\pi/4) \cdot \sin^2(2t+\pi/4)dt + \int_0^{2\pi} \cos(t)\cdot \sin^2(t)dt \\
&= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} u^2 du + \int_0^0 v^2 dv \\
&= 0.
\end{aligned}$$

(Note that we used the two substitutions $u = \sin(3t+\pi/4), v = \sin(t)$ above to evaluate this integral.)