Math 1c	TA: Padraic Bartlett

Homework 8: Stokes' Theorem

Week 9: Due 6/3/13 | *Caltech 2013*

For details on the collaboration policy, due dates, etc., please refer to the Ma1c course webpage. If you have any questions when working on the HW, please don't hesitate to contact your TA (or really any of the TA's,) or indeed even your fellow students!

As always, show your work.

#8.2.3. Suppose that S is the surface $\{(x,y,z) : x^2+y^2+z^2 = 1, z \geq 0\}$, ∂S is $\{(x,y,0) : x^2+y^2 = 1\}$, and $\mathbf{F}$ is the vector field $\mathbf{F}(x,y,z) = (x,y,z)$. Suppose that S is oriented as a graph. Verify Stokes' theorem: i.e. calculate the two integrals

$$\iint_S \nabla \times \mathbf{F} \cdot dS, \quad \int_{\partial S} \mathbf{F} \cdot ds$$

and show that they are equal.

#8.2.5. Repeat the task given in #8.2.3 with $S = \{(x,y,z) : z = 1-x^2-y^2, z \geq 0\}$, $\partial S = \{(x,y,0) : x^2+y^2 = 1\}$, $\mathbf{F}(x,y,z) = (z, x, 2xz+2xy)$. (Assume S is oriented as a graph.)

#8.2.8. Let C be the closed oriented curve formed by traveling in straight lines between $(0,0,0), (2,1,5), (1,1,3)$, and then back to $(0,0,0)$, in that order. Use Stokes' theorem to evaluate

$$\int_C (xyz, xy, x) dc$$

#8.2.10. Using Stokes' theorem, evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot dS$, where $S = \{(x,y,z) : x^2+y^2+z^2 = 16, z \geq 0\}$ and $\mathbf{F}(x,y,z) = (x^2+y-4, 3xy, 2xz+z^2)$. Assume that S is oriented with an upward-pointing normal.

#8.2.13. Let S be the capped cylindrical surface given by taking the union of the two surfaces S_1, S_2, where

$$S_1 = \{(x,y,z) : x^2+y^2 = 1, 0 \leq z \leq 1\},$$
$$S_2 = \{(x,y,z) : x^2+y^2+(z-1)^2 = 1, z \geq 1\}.$$

Set $\mathbf{F}(x,y,z) = (xz+yz^2+x, xyz^3+y, x^2z^4)$. Compute $\iint_S (\nabla \times \mathbf{F}) \cdot dS$ for this surface, using Stokes' theorem. (Assume S is oriented with an outward-pointing normal.)

#8.2.19. Let $\mathbf{F}(x,y,z) = (y, -x, x^3y^2z)$. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$, where S is the surface defined by $x^2+y^2+z^2 = 1, z \leq 0$.

#8.2.24. Take any surface S and any fixed vector $\mathbf{v}$ in $\mathbb{R}^3$. Let $\mathbf{r}(x, y, z) = (x, y, z)$. Prove that

$$2 \iint\limits_{S} \mathbf{v} \cdot \mathbf{n} \, dS = \int\limits_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot ds.$$

#8.2.36. In $\mathbb{R}^3$, one frequently-useful concept we ran into was the idea of **unit vectors** associated with a given coördinate system. In particular, when we worked with the Cartesian coördinate system, we often used the three unit vectors

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$$

and would write things like $xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ instead of the vector (xy, yz, xz).

As it turns out, you can come up with these unit vectors for other coördinate systems as well! This problem asks you to do this same for the spherical coordinate system. Specifically, let $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi$ denote the three orthonormal (i.e. orthogonal and norm 1) vectors associated with the spherical coördinate system. Express each of these as some combination of the three unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

I.e. for a fixed point (ρ, θ, ϕ) in spherical space, find α, β, γ such that

$$\mathbf{e}_\rho = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k},$$

where α, β, γ are determined by the values (ρ, θ, ϕ). Do the same for $\mathbf{e}_\theta, \mathbf{e}_\phi$.