Math 1c	TA: Padraic Bartlett

Recitation 10: Final Review

Week 10	*Caltech 2013*

These are the notes for the final review I ran in my Thursday recitation. In rec, I only got through like 2-3 examples: these notes have all five of the examples I planned.

First, a quick summary of what's going to be on the final. We're testing you on all of the material we've covered since the midterm, i.e. weeks 5-10. A quick list of what we've done in class thus far is listed here:

1. Integration in $\mathbb{R}^n$.
   (a) How to integrate a scalar function over regions in $\mathbb{R}^2$ and volumes in $\mathbb{R}^3$.
   (b) How to integrate a scalar function over surfaces in $\mathbb{R}^3$.
   (c) How to integrate a scalar function over a curve.
   (d) How to integrate a vector field over a surface in either $\mathbb{R}^2$ or $\mathbb{R}^3$.
   (e) How to integrate a vector field over a curve.
2. How to parametrize surfaces and curves.
3. Integration tools.
   (a) Green's theorem.
   (b) Stokes's theorem.
   (c) Divergence (i.e. Gauss's) theorem.
4. Applications of the integral.
   (a) How to use Green's theorem to calculate the area of an object.
   (b) How to use the integral to find the average value of a function, or center of mass of an object.
   (c) How to use Green's/Stokes's theorem to switch between different surface integrals / line integrals.

We've talked about the theory behind these concepts before: if the concepts are still hazy, look them up in previous notes. Instead, what we're going to do here is work example problems on these concepts. If you want to use this to practice for the final, try reading each problem, seeing if you can solve it yourself, and then read the solutions. These are chosen to collectively be about as hard as the problems on the midterm (at least, I think they are.)

1 Practice Questions

Question 1 *Let S denote the cut-off paraboloid surface formed by the equations $z+1 = x^2+y^2$, $z \leq 0$, oriented so that the z-component at the origin is positive. Let F denote the vector field $F(x,y,z) = \left(e^z y, e^{z^2}x, e^{z^3}z\right)$. Find the integral of $\nabla \times F$ over S.*

Solution. First, we calculate $\nabla \times F$:

$$\begin{aligned}\nabla \times F &= \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right) \\ &= \left(0 - 2e^{z^2}xz, e^z y - 0, e^{z^2} - e^z\right).\end{aligned}$$

You could parametrize S and directly integrate this vector over S. But this looks **awful.** Instead, what we can do is use Stokes' theorem! In particular, consider the surface D given by the unit disk $x^2+y^2 \leq 1, z = 0$. This surface has the same boundary as our surface S: specifically, $\partial S = \partial D = x^2+y^2 = 1$. Suppose we orient the unit disk with the normal $(0,0,1)$, which is normal to the unit disk everywhere. Then these boundaries have the same orientation, if both boundaries are oriented positively with respect to their corresponding surfaces.

Therefore, we can use Stokes's theorem once to see that

$$\iint\limits_S \nabla \times F \cdot dS = \int\limits_{\partial S^+} F \cdot ds,$$

and we can use it again to see that

$$\int\limits_{\partial D^+} F \cdot ds = \iint\limits_D \nabla \times F \cdot dS.$$

Because $\partial S = \partial D$, these integrals are all the same! So, to calculate $\iint\limits_S \nabla \times F \cdot dS$, we can calculate $\iint\limits_D \nabla \times F \cdot dS$. We do this here. Notice that the unit normal $\mathbf{n}$ to the unit disk as a surface in $\mathbb{R}^3$ is simply $(0,0,1)$; this saves us the effort of having to parametrize the disk, because

$$\iint\limits_D \nabla \times F \cdot dS = \iint\limits_D \cdot(0,0,1)\ dS = 0.$$

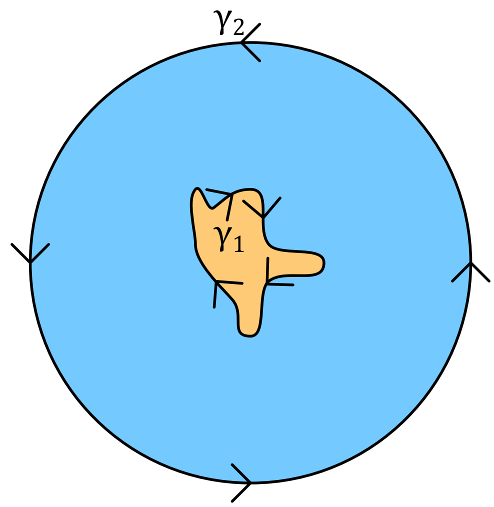
Lots of set-up, but it makes our calculations trivial: we didn't even have to parametrize the unit disk! This is one of the cooler applications of Stokes's theorem: switching between different surfaces.

You can also use things like Stokes's and Green's theorem to switch integrals between different curves: it's a little weirder, but sometimes is really useful.

Question 2 *Take a pond whose outer perimeter is given by a circle of radius 4 and contains 16π cubic centimeters of water. Drop a rock in the center of the pond. Assume that the rock's edges are roughly vertical, i.e. we can model the boundary of the rock in the pond as some 2-d shape. After doing this, assume the water has height h in centimeters.*

Suppose that there is an ant walking around the boundary of the rock. Suppose further that this ant is being blown on by a wind current, which imparts force on the ant corresponding to the vector field $\mathbf{F}(x, y) = (-y, x)$. In one walk of the ant around the boundary of the rock, how much energy does the wind impart on the ant? In other words, what is $\int_{\gamma_1} \mathbf{F} \cdot ds$?

Solution. We draw the situation here.



As labelled above, let γ_1 denote the perimeter of the rock, and γ_2 denote the perimeter of the pond. Let R denote the region between the outer curve and the inner curve. We want to calculate

$$\int\limits_{\gamma_1} \mathbf{F} \cdot dS.$$

This is. . . hard, because, well, we don't actually know what γ_1 **is**. However, we can get around this with Green's theorem!

In particular: notice that Green's theorem says that the integral of $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ over R is equal to the integral of $\mathbf{F}$ over the two boundary components γ_1, γ_2, provided that they're both oriented (as drawn) so that R is always on the left-hand-side of each curve. In other words,

$$\iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_{\gamma_1} \mathbf{F} \cdot ds + \int_{\gamma_2} \mathbf{F} \cdot ds.$$

So, we can solve for the integral we want to study, in terms of two other integrals:

$$\int_{\gamma_1} \mathbf{F} \cdot ds = \iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA - \int_{\gamma_2} \mathbf{F} \cdot ds.$$

These are, surprisingly, things we can calculate. In specific, we have that

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R (1-(-1))\, dA = \iint_R 2dA = 2 \cdot (\text{surface area of } R).$$

Because the pond started with 16π cubic centimeters of water and had height h after we dropped the rock in, we know that R has surface area $\frac{16\pi}{h}$, and therefore that this integral is $\frac{32\pi}{h}$.

As well, we can find $\int_{\gamma_2} \mathbf{F} \cdot ds$. We parametrize γ_2 as $\gamma_2(t) = (4\cos(t), 4\sin(t))$:

$$\int_{\gamma_2} \mathbf{F} \cdot ds = \int_0^{2\pi} (-4\sin(t), 4\cos(t)) \cdot (-4\sin(t), 4\cos(t))\, dt = \int_0^{2\pi} 16dt = 32\pi.$$

Therefore, we can combine these two integrals to calculate $\int_{\gamma_1} \mathbf{F} \cdot ds$:

$$\int_{\gamma_1} \mathbf{F} \cdot ds = 32\pi \left(\frac{1}{h} - 1\right).$$

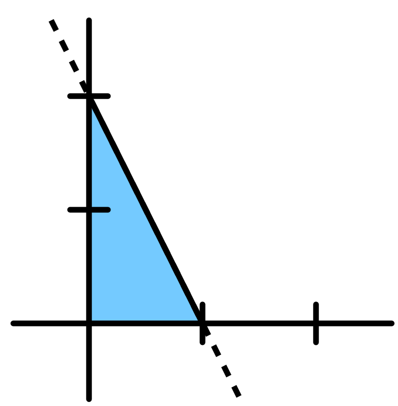
This is pretty cool: we know exactly how much work was done by this wind current, even though we have no idea what path we integrated over!

Question 3 *Let T be a triangle with vertices $(1,0,0),(0,2,0),(0,0,3)$. If this triangle is made out of some material with uniform density across its surface, what is the x-coördinate of the center of mass of T?*

Solution. We want to find the x-coördinate of the center of mass of T. This is the "average" x-coördinate over our entire surface. Recall the following: if we want to find the average value of a function f on a surface T, we want to find the integrals $\iint_T f dA$ and $\iint_T 1 dA$, and divide the first of these two integrals by the second: this gives us the average value of f over T.

So. We start by parametrizing our triangle. We do this by considering coördinates one-by-one. We first look at x: over our entire triangle, x ranges from 0 to 1.

We now look at the possible range of y-values, given x. We do this by projecting our triangle onto the xy-plane: there, this is the triangle with vertices $(0,0),(1,0),(0,2)$.



Given any fixed value of x, we can see that y ranges from 0 to $2-2x$.

Finally, we need to solve for z given x and y. To do this, we just need to find the plane this triangle lies in: this will give us an equation relating x, y and z. We do this by taking the generic equation for a plane

$$ax + by + cz = d$$

and plugging in the three points $(1,0,0), (0,2,0), (0,0,3)$ into this equation:

$$a = d, b = \frac{d}{2}, c = \frac{d}{3}.$$

This gives us that our plane has the equation

$$dx + \frac{d}{2}y + \frac{d}{3}z = d,$$

which (if we divide by d) becomes

$$x + \frac{y}{2} + \frac{z}{3} = 1.$$

Solving for z gives us

$$z = 3 - 3x - \frac{3y}{2}.$$

So, we can parametrize our triangle via the map $T(x,y) = \left(x, y, 3 - 3x - \frac{3y}{2}\right)$, where x ranges from 0 to 1 and (given x) y ranges from 0 to $2-2x$.

So, if we want to find $\iint_T 1 dA$, we can just use this parametrization:

$$\begin{aligned}
\iint_T 1\ dA &= \int_0^1 \int_0^{2-2x} \left|\left|\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y}\right|\right| dxdy \\
&= \int_0^1 \int_0^{2-2x} \left|\left|\left(1, 0, -3\right) \times \left(0, 1, -\frac{3}{2}\right)\right|\right| dxdy \\
&= \int_0^1 \int_0^{2-2x} \left|\left|\left(3, \frac{3}{2}, 1\right)\right|\right| dxdy \\
&= \int_0^1 \int_0^{2-2x} \sqrt{9 + \frac{9}{4} + 1}\ dxdy \\
&= \int_0^1 \int_0^{2-2x} \frac{7}{2}\ dxdy \\
&= \int_0^1 7 - 7x\ dx \\
&= \frac{7}{2}.
\end{aligned}$$

Similarly, if we want to find $\iint_T x dA$, we can do mostly the same thing:

$$\begin{aligned}\iint_T 1\ dA &= \int_0^1 \int_0^{2-2x} x \left|\left|\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y}\right|\right| dxdy \\ &= \int_0^1 \int_0^{2-2x} \frac{7x}{2}\ dxdy \\ &= \int_0^1 7x - 7x^2\ dx \\ &= \frac{7}{6}.\end{aligned}$$

Therefore, the x-coördinate of the center of mass is just the ratio of these two integrals, i.e. $\frac{7/6}{7/2} = \frac{1}{3}$.

Question 4 *Let T be the same triangle as in Question 3. Integrate the vector field $\mathbf{F}(x, y, z) = (xy, yz, zx)$ over the perimeter of this triangle, oriented in the counterclockwise direction as viewed from the positive octant.*

Solution. We could parametrize the boundary of this triangle, but that seems hard. Instead, we will use Stokes's theorem, which says that

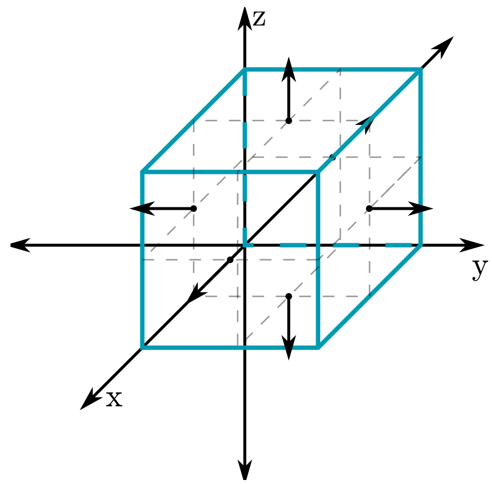
$$\int\limits_{\partial T} \mathbf{F} \cdot ds = \iint\limits_T \nabla \times \mathbf{F} \cdot dS.$$

Using this, we can instead integrate $\nabla \times \mathbf{F}$ over the triangle itself, because we already parametrized that! Convenient.

We do this here.

$$\begin{aligned}\iint\limits_T \nabla \times \mathbf{F} \cdot dS &= \int_0^1 \int_0^{2-2x} (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y}\right) dA \\ &= \int_0^1 \int_0^{2-2x} \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right) \cdot \left(\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y}\right) dA \\ &= \int_0^1 \int_0^{2-2x} (0 - y, 0 - z, 0 - x)\Big|_{T(x,y)} \cdot \left(3, \frac{3}{2}, 1\right) dA \\ &= \int_0^1 \int_0^{2-2x} -3y - \frac{3}{2}\left(3 - 3x - \frac{3}{2}y\right) - x\ dA \\ &= \int_0^1 \int_0^{2-2x} -\frac{9}{2} - \frac{3}{4}y + \frac{7}{2}x\ dA \\ &= \int_0^1 -9 + 9x - \frac{3}{8}(2 - 2x)^2 + 7x - 7x^2\ dA \\ &= \int_0^1 -\frac{17}{2}x^2 + 19x - \frac{21}{2}\ dA \\ &= -\frac{17}{6} + \frac{19}{2} - \frac{21}{2} = -\frac{23}{6}.\end{aligned}$$

Question 5 *Directly calculate the integral of* $F(x, y, z) = (3x^2y, -3xy^2, z)$ *over the surface of the unit cube, using the orientation depicted below. Then, use the divergence theorem to calculate this in a much faster manner.*



Solution. If we want to do this directly, break the unit cube into its six sides

$$[0,1] \times [0,1] \times \{0\}, [0,1] \times [0,1] \times \{1\},$$
$$[0,1] \times \{0\} \times [0,1], [0,1] \times \{1\} \times [0,1]$$
$$\{0\} \times [0,1] \times [0,1], \{1\} \times [0,1] \times [0,1],$$

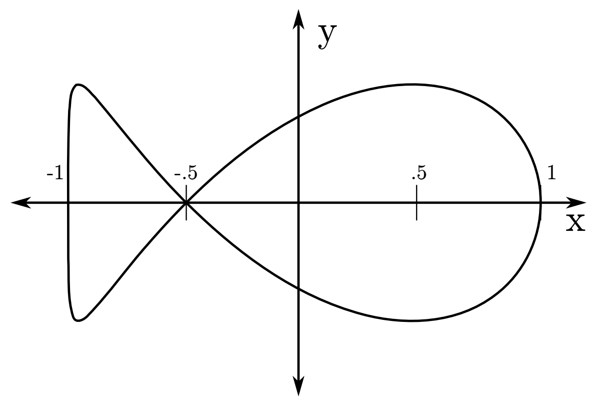
notice that the normals to these sides are precisely the normals $(0,0,\pm1), (0,\pm1,0), (\pm1,0,0)$ as depicted in the above diagram, and calculate

$$\begin{aligned}
&\iint_{\text{surface of cube}} F \cdot dS \\
=&\int_0^1 \int_0^1 F\big|_{(x,y,0)} \cdot (0,0,-1)dxdy + \int_0^1 \int_0^1 F\big|_{(x,y,1)} \cdot (0,0,1)dxdy \\
&\quad + \int_0^1 \int_0^1 F\big|_{(x,0,z)} \cdot (0,-1,0)dxdz + \int_0^1 \int_0^1 F\big|_{(x,1,z)} \cdot (0,1,0)dxdz \\
&\quad + \int_0^1 \int_0^1 F\big|_{(0,y,z)} \cdot (-1,-0,0)dydz + \int_0^1 \int_0^1 F\big|_{(1,y,z)} \cdot (1,0,0)dydz \\
=&\int_0^1 \int_0^1 0dxdy + \int_0^1 \int_0^1 1dxdy + \int_0^1 \int_0^1 0dxdz + \int_0^1 \int_0^1 -3x\ dxdz \\
&\quad + \int_0^1 \int_0^1 0dydz + \int_0^1 \int_0^1 3ydydz \\
=&1.
\end{aligned}$$

Alternately, if you use the divergence theorem, we can calculate this in a much faster way:

$$\iint_{\text{surface of cube}} F \cdot dS = \iiint_{\text{cube}} (\text{div}F) dV$$
$$= \int_0^1 \int_0^1 \int_0^1 (6xy - 6xy + 1) dxdydz$$
$$= \int_0^1 \int_0^1 \int_0^1 1 dxdydz = 1.$$

Question 6 *Let* $c(t) = \left(\cos(t) - \frac{\sin^2(t)}{2}, \cos(t)\sin(t)\right)$ *denote the "fish curve" drawn below:*



Find the area contained within this curve.

Solution. This looks like a textbook example of when to use the Green's theorem formula for the area contained in a curve. Specifically, Green's theorem, as applied to finding the area contained within a curve, says that if a region R is bounded by some simple closed curve $c(t)$ that is oriented positively (i.e. so that R is on the left as we travel along $c(t)$), then

$$\text{area}(R) = \iint_R 1 dxdy \overbrace{=}^{\text{Green's theorem}} = \frac{1}{2} \int_{c(t)} (-y, x)\, dc.$$

If we just plug in our curve, we get that $\frac{1}{2}\int_{c(t)}(-y,x)\,dc$ is

$$
\begin{aligned}
&\frac{1}{2}\int_0^{2\pi}\left(-\cos(t)\sin(t),\cos(t)-\frac{\sin^2(t)}{2}\right)\cdot\left(-\sin(t)-\sin(t)\cos(t),\cos^2(t)-\sin^2(t)\right)dt\\
=&\frac{1}{2}\int_0^{2\pi}\left(\cos(t)\sin^2(t)+\cos^2(t)\sin^2(t)+\cos^3(t)-\cos(t)\sin^2(t)-\frac{\cos^2(t)\sin^2(t)}{2}+\frac{\sin^4(t)}{2}\right)dt\\
=&\frac{1}{2}\int_0^{2\pi}\left(\frac{\cos^2(t)\sin^2(t)}{2}+\cos^3(t)+\frac{\sin^4(t)}{2}\right)dt\\
=&\frac{1}{2}\int_0^{2\pi}\left(\frac{\sin^2(2t)}{8}+\cos(t)(1-\sin^2(t))+\frac{(1-\cos(2t))^2}{8}\right)dt\\
=&\frac{1}{2}\int_0^{2\pi}\left(\frac{1-\cos(4t)}{16}+\cos(t)(1-\sin^2(t))+\frac{1-2\cos(2t)+\cos^2(2t)}{8}\right)dt\\
=&\frac{1}{2}\int_0^{2\pi}\left(\frac{1-\cos(4t)}{16}+\cos(t)(1-\sin^2(t))+\frac{1-2\cos(2t)}{8}+\frac{1+\cos(4t)}{16}\right)dt\\
=&\frac{1}{2}\int_0^{2\pi}\left(\frac{1}{4}+\cos(t)(1-\sin^2(t))-\frac{\cos(2t)}{4}\right)dt\\
=&\frac{1}{2}\left(\frac{t}{4}+\sin(t)-\frac{\sin^3(t)}{3}-\frac{\sin(2t)}{8}\right)\Big|_0^{2\pi}\\
=&\pi/4.
\end{aligned}
$$

But is this plausible? Well: looking at our fish curve, it seems to contain about (in the head-part) the area of an ellipse from $-.5$ to 1 with y-height from -1 to 1, which is about $3\pi/2$. This is much greater than $\pi/4$, the area of a circle with radius .5. So: something has gone wrong!

What, specifically? Well, to apply Green's theorem, we needed a **simple closed curve** that was positively oriented. Did we have that here? No! In fact, our curve c has a self-intersection: $c(\pi/2)=c(3\pi/2)$, and in fact the tail part of our curve is oriented negatively (i.e. if we travel around our curve from $\pi/2$ to $3\pi/2$, our region is on the right-hand side. In fact, we've calculated the area of the head minus the area in the tail!

To calculate what we want, we want to take the integral above evaluated from $-\pi/2$ to $\pi/2$ (the head) and then add the integral from $3\pi/2$ to $\pi/2$ (travelling backwards here makes it so that we get the right orientation on the tail.) Specifically, we have

$$
\begin{aligned}
\frac{1}{2}\left(\frac{t}{4}+\sin(t)-\frac{\sin^3(t)}{3}-\frac{\sin(2t)}{8}\right)\Big|_{-\pi/2}^{\pi/2}=&\frac{1}{2}\left(\frac{\pi}{8}-\frac{-\pi}{8}+1-(-1)+\left(-\frac{1}{3}\right)-\frac{1}{3}+0-0\right)\\
=&\frac{\pi}{8}+\frac{2}{3},
\end{aligned}
$$

while

$$
\begin{aligned}
\frac{1}{2}\left(\frac{t}{4}+\sin(t)-\frac{\sin^3(t)}{3}-\frac{\sin(2t)}{8}\right)\Big|_{3\pi/2}^{\pi/2}=&\frac{1}{2}\left(-\frac{\pi}{8}-\frac{\pi}{8}+1-(-1)+\left(\frac{-1}{3}\right)-\frac{1}{3}+0-0\right)\\
=&-\frac{\pi}{8}+\frac{2}{3};
\end{aligned}
$$

therefore, our total area is $\frac{\pi}{8}+\frac{2}{3}+-\frac{\pi}{8}+\frac{2}{3}=\frac{4}{3}$.