Math 1c	TA: Padraic Bartlett
Recitation 7: Integrals on Surfaces	
Week 7	*Caltech 2013*

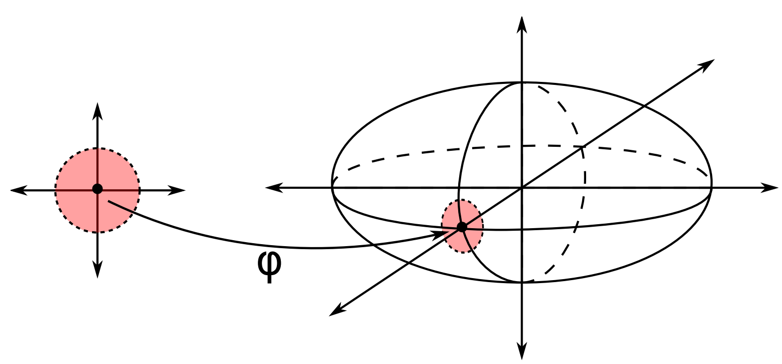
1 Surfaces

In Math 1 this year, we've described lots of things as "surfaces," and used the concept several times in Ma1b and Ma1c when describing objects and setting up problems. So, um, something we should do if we want to keep using this concept is actually **define** what a surface is! We explore this here.

To get an idea of what we might want one to be, let's examine several things we **want** to be surfaces: spheres, tori, cones, paraboloids, sheets, cubes, and the graphs of continuous functions $z = f(x, y)$. What do these all have in common? Well, intuitively speaking, they all "locally look like $\mathbb{R}^2$ — i.e. if you pick a point on a sphere, or a point on a torus, or on a plane, and zoom in really really close, it looks like a tiny piece of the plane!

As it turns out, this notion, of "locally looking like $\mathbb{R}^2$," is an excellent candidate for the definition of a surface. In the following definition, we make this notion rigorous:

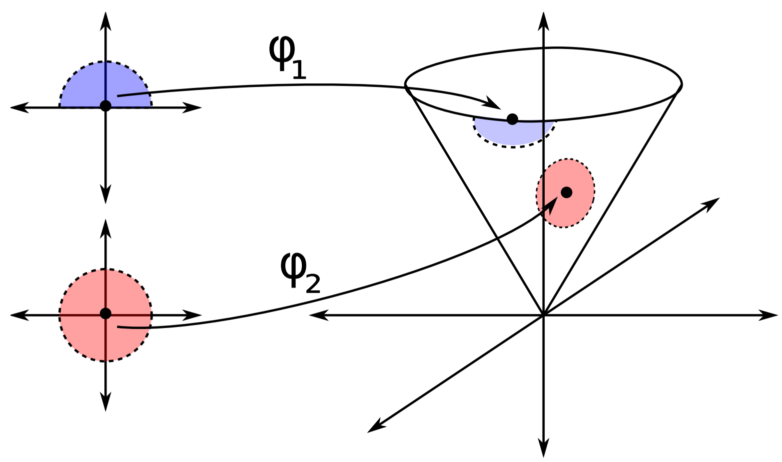
Definition. A subset $S \subseteq \mathbb{R}^n$ is called a **surface without boundary** if for every point $s \in S$, there is an open neighborhood N_s of s and a continuous, 1-1 and onto function φ from the open unit disk $\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ to the set $N_s \cap S$. In other words, for every point s in S, there is a little neighborhood of s in which S locally looks like $\mathbb{R}^2$.



Similarly, a surface $S \subseteq \mathbb{R}^n$ is called a **surface with boundary** if for every point $s \in S$, we have one of the following two cases:

1. There is an open neighborhood N_s of s and a continuous, 1-1 and onto function $\varphi_1 : \{(x, y) : x^2 + y^2 < 1, y \geq 0\} \to N_s$. In this case, s is a boundary point of S.

2. There is an open neighborhood N_s of s and a continuous, 1-1 and onto function $\varphi_2 : \mathbb{D} \to N_s$. In this case, s is an interior point of S.

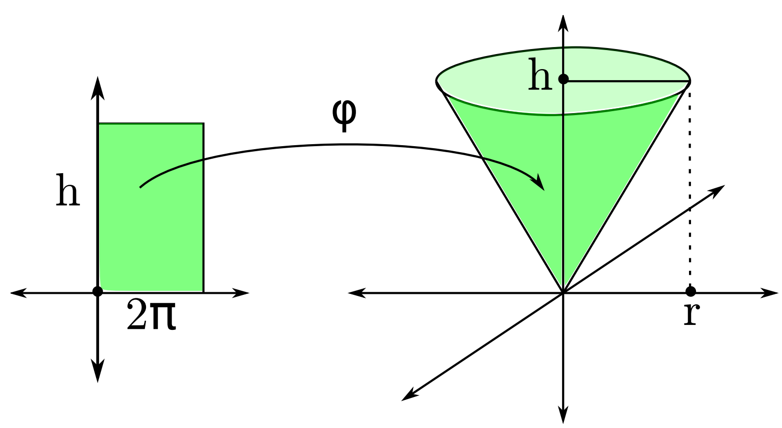


One frustrating thing about this definition is that it only gives us these maps φ locally. There are many situations where we'd like to not have to deal with this issue: i.e. instead of having to deal with a bunch of different maps, we'd like say **one map**[1] that makes everything on our surface S look like $\mathbb{R}^2$.

Definition. We say that S is a **surface parametrized by** φ if and only if there is a region $R \subset \mathbb{R}^2$ and associated continuous onto function $\varphi : R \to S$, that's one-to-one except perhaps on the boundary points of R.

This definition is perhaps best illustrated by a series of examples of parametrized surfaces:

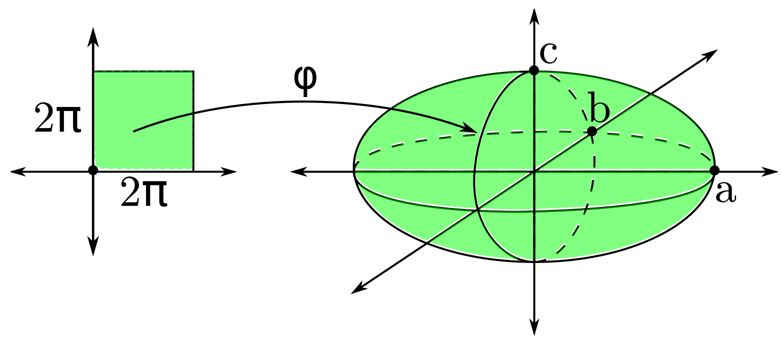
Example. A cone of height h and radius r around the z-axis, with apex at $(0,0)$ as depicted below, can be parametrized by the map $\varphi : [0,h] \times [0,2\pi] \to \mathbb{R}^3$, $\varphi(z,\theta) = \left(\frac{z \cdot r}{h}\cos(\theta), \frac{z \cdot r}{h}\sin(\theta), z\right)$.



If you want to double-check this, simply use cylindrical coördinates to see that the image of the set above is indeed a cone!

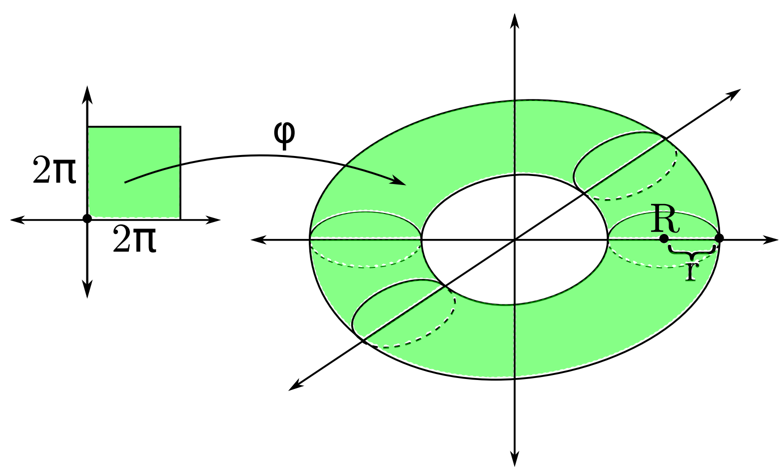
[1] To rule them all?

Example. A ellipsoid that intersects the x-axis at a, y-axis at b, and z-axis at c, as depicted below, can be parametrized by the map $\varphi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$, $\varphi(\phi, \theta) = (a \sin(\phi) \cos(\theta), b \sin(\phi) \sin(\theta), c \cos(\phi))$.



Similarly to the above, you can double-check that this is valid by using spherical coördinates.

Example. A torus around the circle $x^2 + y^2 = R^2$, with internal radius r (as depicted below) can be parametrized by the map $\varphi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$, with $\varphi(\phi, \theta) = (\cos(\phi)(R + r\cos(\theta)), \sin(\phi)(R + r\cos(\theta)), r\sin(\theta))$.



1.1 Integrals on surfaces.

As this is a calculus class, the natural question to ask (when given any new object) is "How can we integrate over this?" In other words, suppose we have a surface $S \subset \mathbb{R}^3$, and some function $f : \mathbb{R}^3 \to \mathbb{R}$. What would we possibly mean by the **integral** of f on S?

Well: suppose for the moment that f is parametrized by some function $\varphi : R \to S$, $R \subseteq \mathbb{R}^2$. Then, one natural way to define the integral of f over S is to say that it is the integral of $f \circ \varphi$ over R, where we need to compensate for how φ "stretches areas." To be explicit, we have the following definition:

Definition. For a surface $S \subset \mathbb{R}^3$ parametrized by some function $\varphi(x, y) : R \to S$, $R \subseteq \mathbb{R}^2$ and some function $f : \mathbb{R}^3 \to \mathbb{R}$, we define the **integral** of f over S as the following expression:

$$\iint_S f\ dS = \iint_R f(\varphi(x,y)) \cdot \left|\left|\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y}\right|\right| dxdy.$$

The $\left|\left|\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y}\right|\right|$ bit above, specifically, is the thing that corrects for how φ distorts space. Specifically, at any point (x, y), it's distorting space by $\frac{\partial \varphi}{\partial x}$ as you increase x slightly and by $\frac{\partial \varphi}{\partial y}$ as you increase y: therefore, it's distorting area by the magnitude of the cross-product of those two vectors at that point!

1.2 Example calculations.

To demonstrate how these concepts work, we calculate two examples:

Example. Calculate the surface area of a cone C with height 1 and radius 1 (using the height and radius notation from our earlier parametrizations.)

Solution. First, note that the surface area of any surface S is just the integral of the function 1 over the entire surface: therefore, this problem is just asking us to find $\iint_S 1\ dS$.

Let $\varphi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$, $\varphi(z, \theta) = (z \cos(\theta), z \sin(\theta), z)$ be the parametrization of the cone we discussed earlier in recitation. Then, by definition, we have that

$$\begin{aligned}
\iint_C 1\ dS &= \int_0^1 \int_0^{2\pi} 1 \cdot \left|\left|\frac{\partial \varphi}{\partial z} \times \frac{\partial \varphi}{\partial \theta}\right|\right| d\theta dz \\
&= \int_0^1 \int_0^{2\pi} 1 \cdot ||(\cos(\theta), \sin(\theta), 1) \times (-z\sin(\theta), z\cos(\theta), 0)||\, d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \left|\left|(-z\cos(\theta), -z\sin(\theta), z\cos^2(\theta) + z\sin^2(\theta))\right|\right| d\theta dz \\
&= \int_0^1 \int_0^{2\pi} ||(-z\cos(\theta), -z\sin(\theta), z)||\, d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \sqrt{(-z\cos(\theta))^2 + (z\sin(\theta))^2 + z^2 d\theta dz} \\
&= \int_0^1 \int_0^{2\pi} \sqrt{z^2\cos^2(\theta) + z^2\sin^2(\theta)2 + z^2 d\theta dz} \\
&= \int_0^1 \int_0^{2\pi} \sqrt{2z^2} d\theta dz \\
&= \int_0^1 \int_0^{2\pi} z\sqrt{2} d\theta dz \\
&= \int_0^1 2\pi z\sqrt{2} dz \\
&= \pi\sqrt{2}.
\end{aligned}$$

Example. Find the center of mass of a cone C centered on the z-axis of height 1 and radius 1, if it has uniform area density 1 (i.e. its area density function is $\gamma(x, y, z) = 1$.)

Solution. (Recall that a **center of mass** for any object is a point (x, y, z) such that any plane cutting through (x, y, z) will have half of the object's mass on either side of this plane. Also, recall that we can find this by finding the *average* of each of the coördinates x, y, z over this surface, weighted by the density function $\gamma(x, y, z)$: in other words, the x-coördinate of the center of mass of any surface with density function γ is just the quantity $\iint_S x \cdot \gamma dS$ divided by the total mass of S, $\iint_S \gamma dS$.

First, notice that any such cone centered on the z-axis must have the x- and y-coördinates of its center of mass both be zero, as this cone is symmetric around the x- and y-axes.

So, it suffices to find the z-coördinate of the center of mass of our cone. To do this, because our density function is identically 1, we just need to find the integral

$$\iint_C z\gamma \; dS = \iint_C z \; dS$$

and divide it by the total mass of the cone,

$$\iint_C \gamma \; dS = \iint_C 1 \; dS = \pi\sqrt{2}$$

(from our above calculations.)

Using the definition of the integral, and our earlier calculation that $\left|\left|\frac{\partial \varphi}{\partial z} \times \frac{\partial \varphi}{\partial \theta}\right|\right| = z\sqrt{2}$, we have that

$$\begin{aligned}
\iint_C z \; dS &= \int_0^1 \int_0^{2\pi} z \cdot \left|\left|\frac{\partial \varphi}{\partial z} \times \frac{\partial \varphi}{\partial \theta}\right|\right| d\theta dz \\
&= \int_0^1 \int_0^{2\pi} z^2\sqrt{2} d\theta dz \\
&= \int_0^1 2\pi z^2 \sqrt{2} dz \\
&= 2\pi\sqrt{2}/3.
\end{aligned}$$

So, the z-coördinate of our center of mass is just

$$\frac{2\pi\sqrt{2}/3}{\pi\sqrt{2}} = 2/3.$$

So, if you have a right cone with uniform density (say, you made your cone out of paper), and you puncture it about 2/3-rds of the way up with a pencil or dowel or somesuch thing, it should spin freely about that axis, as its weight is equally distributed on all sides.