Math 1c	TA: Padraic Bartlett
Recitation 9: Green's Theorem	
Week 9	*Caltech 2013*

1 Green's Theorem: Motivation, Statement and Examples

Today's lecture, like almost every lecture we've given this quarter, is about how we can extend a concept from one-dimensional calculus to higher dimensions. Throughout this course, we've already extended the concepts of limits, derivatives, several derivative techniques, integrals, and several integral techniques from $\mathbb{R}^1$ to $\mathbb{R}^n$; basically, whenever we've seen anything in single-variable calculus, we've been able to extend it to $\mathbb{R}^n$. Loosely speaking, there's really only one major theorem that we haven't extended yet: the **Fundamental Theorem of Calculus**, which stated that (for $f : \mathbb{R} \to \mathbb{R}$ a C^1 function)

$$\int_a^b \frac{d}{dx}(f(x))dx = f(b) - f(a).$$

In other words, knowing the behavior of the derivative over an interval is equivalent to knowing the function's original values at the endpoints of that interval. This, you may remember, was a remarkably powerful technique: in single-variable calculus, the FTC often allowed us to transform knowledge of the derivative (often a far simpler thing than the original function) over a region into the function's actual behavior on the boundary of this region, and vice-versa.

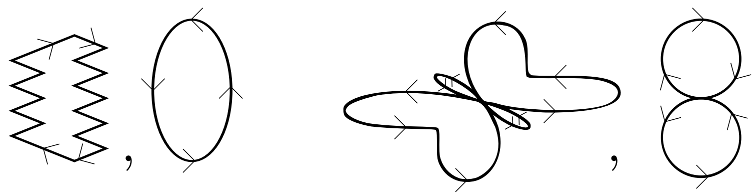
A natural question to ask, then, is whether we can extend this to higher dimensions. I.e. take a region $R \subset \mathbb{R}^2$, with boundary ∂R. Can we relate the behavior of a function on ∂R to the behavior of some sort of derivative on all of R?

As it turns out, we can! This is precisely Green's theorem; to state it formally, we first make the following two definitions.

Definition. A **simple closed curve** γ is a map $[a, b] \to \mathbb{R}^n$ such that

- $\gamma(a) = \gamma(b)$,
- γ has finite length, and
- γ does not intersect itself: i.e. for any two points $x \neq y \in [a, b]$, $\gamma(x) = \gamma(y)$ if and only if x and y are the two endpoints a, b.

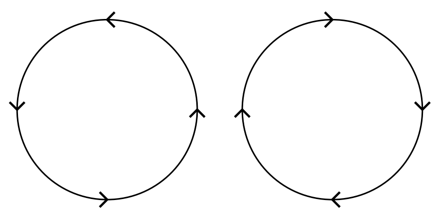
Example. The following illustrates some closed curves that are simple, and some closed curves that are not simple:



(simple closed curves) (not simple closed curves)

Definition. Suppose that a simple closed curve γ is also the boundary of some region R. We say that a curve is **positively oriented** if travelling along our curve in the direction given by γ keeps R on the "left" of the curve. Similarly, a parametrization is **negatively oriented** if travelling along the curve keeps R on the "right."

Example. For example, the parametrization $\gamma_+(t) = (\cos(t), \sin(t))$ is a positively-oriented parametrization with respect to the unit disk. This is because moving along the unit disk using γ keeps the unit disk always on our left. Similarly, the parametrization $\gamma_-(t) = (\cos(t), -\sin(t))$ is negatively-oriented, because the unit disk is always on the right of our parametrization.



(positive) (negative)

Theorem 1 *(Green's Theorem.) Suppose that R is some region in $\mathbb{R}^2$ such that R's boundary is given by the curve C_1, and that γ is a positive parametrization of c_1. Suppose that P and Q are a pair of maps $\mathbb{R}^2 \to \mathbb{R}$ with continuous partial derivatives in an open neighborhood of R. Then, we have the following equality*

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_\gamma Pdx + Qdy$$

2 Green's Theorem: Applications

Why do we care about Green's theorem? Well: from looking at its statement above, what does it do? It takes a pair of functions P, Q and sends an integral involving them to an integral involving their partials $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$; as well, it transforms a line integral over some curve C into a integral over some region R. This suggests that we might want to use Green's theorem in the following situations:

1. If we're integrating a pair of functions over some particularly awful curve, we might want to use Green's theorem to transform this integral into one over a region, in the hopes that the expression $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ might become zero or at the least a simpler expression.

2. Conversely, if we have a fairly awful region R, we might want to use Green's theorem to take us to a line integral, which can sometimes make our lives easier. One typical example of this is the use of Green's theorem to calculate the **area** of a region, which is the following equation:

$$\iint_R 1 \; dxdy = \frac{1}{2} \oint_C xdy - ydx.$$

The left-hand side is (by definition) the area of the region R; the right-hand side is one possible pair of functions P, Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is 1.

We illustrate these two uses with two examples:

Example. For any two constants $a, b \in \mathbb{R}$, and $n \in \mathbb{N}$, find the integral

$$\oint_{C_n^+} a\cos(x)dx + b\sin(y)dy,$$

where C_n^+ is a counterclockwise-oriented n-gon with side length 1, center at (0,0), and one vertex on the x-axis.

Solution. So: this is (clearly) a case where our curve C_n^+ is far too awful to integrate along. Having no other option, we apply Green's theorem, which tells us that (if R is the region enclosed by our n-gon)

$$\begin{aligned}\oint_{C_n^+} adx + bdy &= \iint_R \left(\frac{\partial(b\cos(y))}{\partial x} - \frac{\partial(a\sin(x))}{\partial y} \right) dxdy \\ &= \iint_R (0-0)\, dxdy \\ &= 0.\end{aligned}$$

Done!

Example. Find the area of the ellipse

$$R = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}.$$

Solution. As mentioned before, the area of any region R can be given by the integral $\iint_R 1 \; dxdy$; so, if we choose $P(x,y) = -y/2, Q(x,y) = x/2$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, and thus that

$$\iint_R 1 \; dxdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \frac{1}{2} \oint_{C^+} xdy - ydx,$$

where C^+ is the boundary curve of our ellipse: i.e. $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $\gamma(t) = (a\cos(t), b\sin(t))$.

Calculating, we have

$$\begin{aligned}
\frac{1}{2}\oint_{C^+} xdy - ydx &= \frac{1}{2}\int_0^{2\pi} (-y, x)\Big|_{\gamma(t)} \cdot \gamma'(t)dt \\
&= \frac{1}{2}\int_0^{2\pi} (-b\sin(t), a\cos(t)) \cdot (-a\sin(t), b\cos(t))dt \\
&= \frac{1}{2}\int_0^{2\pi} ab(\sin^2(t) + \cos^2(t))dt \\
&= \frac{1}{2}\int_0^{2\pi} ab\ dt \\
&= ab\pi.
\end{aligned}$$

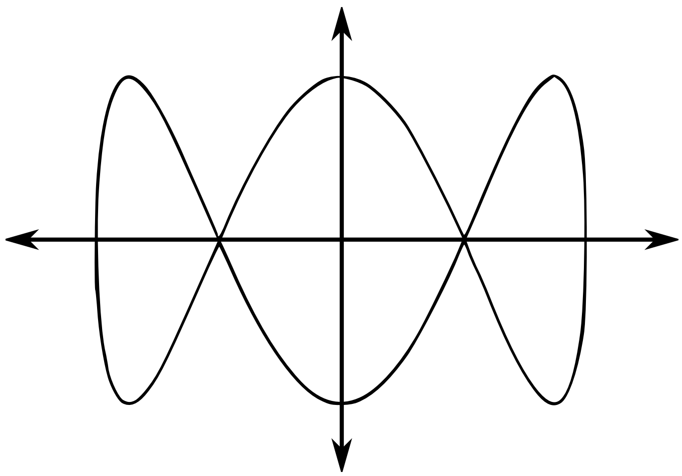
It bears noting that we had many possible choices of P, Q above! Specifically, we could have also chosen $Q = x, P = 0$; in this case, we would have had

$$\begin{aligned}
\iint_R 1\ dxdy &= \oint_{C^+} xdy \\
&= \int_0^{2\pi} (0, a\cos(t)) \cdot (-a\sin(t), b\cos(t))dt \\
&= \int_0^{2\pi} ab\cos^2(t)dt \\
&= \int_0^{2\pi} ab\frac{\cos(2t)+1}{2}dt \\
&= \left(ab\frac{\sin(2t)}{4} + \frac{abt}{2}\right)\Bigg|_0^{2\pi} \\
&= ab\pi.
\end{aligned}$$

This is the same answer! This is just an aside, to illustrate that you can have many different choices of P, Q available to you such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is equal to your desired expression.

The following example provides a slightly tricker area calculation, as well as a cautionary tale about making sure to always check your boundary conditions when you're applying a theorem:

Example. Find the area of the region R enclosed by the Lissajous curve $\gamma(t) = (\cos(t), \sin(3t))$, where t ranges from 0 to 2π.



Solution. When presented with a region R enclosed by a curve γ, it's really tempting to simply directly apply our Green's theorem for area result, which says that when γ is a simple closed curve oriented counterclockwise, we have

$$\text{area}(R) = \iint_R 1 dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma.$$

However, if we just directly apply this here, we'll get that

$$\begin{aligned}\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma &= \int_0^{2\pi} \left(-\frac{\sin(3t)}{2}, \frac{\cos(t)}{2}\right) \cdot (-\sin(t), 3\cos(3t)) dt \\ &= \frac{1}{2} \int_0^{2\pi} \sin(3t)\sin(t) + 3\cos(3t)(\cos(t) dt.\end{aligned}$$

By applying your angle-addition formulas

- $\cos(3t) = \cos(t)\cos(2t) - \sin(t)\sin(2t)$,
- $\sin(3t) = \sin(t)\cos(2t) + \sin(2t)\cos(t)$,

along with your double-angle formulas, we have that this is

$$\begin{aligned}\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma &= \frac{1}{2}\int_0^{2\pi} \sin(t)(\sin(t)\cos(2t) + \sin(2t)\cos(t)) + 3\cos(t)(\cos(t)\cos(2t) - \sin(t)\sin(2t)) dt \\ &= \frac{1}{2}\int_0^{2\pi} \sin^2(t)\cos(2t) + \sin(2t)\sin(t)\cos(t) + 3\cos^2(t)\cos(2t) - 3\sin(t)\cos(t)\sin(2t)) dt\end{aligned}$$

$$= \frac{1}{2}\int_0^{2\pi} \sin^2(t)\cos(2t) + 3\cos^2(t)\cos(2t) + \frac{\sin^2(2t)}{2} - \frac{3\sin^2(2t)}{2}dt$$
$$= \frac{1}{2}\int_0^{2\pi} \cos(2t) + 2\cos^2(t)\cos(2t) - \sin^2(2t)dt$$
$$= \frac{1}{2}\int_0^{2\pi} \cos(2t) + (1+\cos(2t))\cos(2t) - \frac{1-\cos(4t)}{2}dt$$
$$= \frac{1}{2}\int_0^{2\pi} 2\cos(2t) + \frac{1+\cos(4t)}{2} - \frac{1-\cos(4t)}{2}dt$$
$$= \frac{1}{2}\int_0^{2\pi} 2\cos(2t) + \cos(4t)dt$$
$$= 0.$$

Um. So, this is clearly false: our curve, by visual inspection, contains more area than 0. What went wrong? Well, our curve γ is **not** a simple closed curve: it has self-intersections!

So: to fix that, we can break up our curve γ into three parts:

- The part where γ's parameter t is restricted to the set $[-\pi/3, \pi/3]$. This is the far-right part of our curve; here, γ is counterclockwise-oriented, and we can thus find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos(2t) + \cos(4t)dt = \left.\frac{\sin(2t) + \sin(4t)/4}{2}\right|_{-\pi/3}^{\pi/3} = \frac{3\sqrt{3}}{8}.$$

- The part where γ's parameter t is restricted to the set $[4\pi/3, 5\pi/3]$. This is the far-left part of our curve; here, γ is also counterclockwise-oriented, and we can thus find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{4\pi/3}^{5\pi/3} 2\cos(2t) + \cos(4t)dt = \left.\frac{\sin(2t) + \sin(4t)/4}{2}\right|_{4\pi/3}^{5\pi/3} = \frac{3\sqrt{3}}{8}.$$

- The part where γ's parameter t is restricted to the set$[\pi/3, 2\pi/3] \cup [4\pi/3, 5\pi/3]$. Here, γ is clockwise-oriented! Therefore, to find the area enclosed by gamma, we need to take the negative of this signed area, which is

$$\frac{1}{2}\int_{\pi/3}^{2\pi/3} 2\cos(2t) + \cos(4t)dt + \frac{1}{2}\int_{4\pi/3}^{5\pi/3} 2\cos(2t) + \cos(4t)dt = \ldots = \frac{\sqrt{3}}{4}.$$

  Notice that we've used a curve γ here that was piecewise defined: this is completely OK! The only thing you need to check is that the curve is a simple closed one and counterclockwise-oriented: once you've done that, it can be defined however you like.

Summing these three parts gives us that the area enclosed by our curve is $3\sqrt{3}/2$.