Math 1c	TA: Padraic Bartlett
Recitation 9: Stokes' Theorem	
Week 9	*Caltech 2013*

Hey! Remember Green's theorem? Wouldn't it be great if there was a version of Green's theorem that worked in three dimensions?

1 Stokes' Theorem: Motivation and Statement

Fortunately, there is!

Theorem 1 *(Stokes' theorem.) Suppose that S is a bounded surface*[1] *with boundary given by the counterclockwise-oriented curve C, and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is some continuously differentiable function. Then, we have the following equality:*

$$\iint_S ((\nabla \times \mathbf{F}) \cdot \mathbf{n})\, dS = \int_C \mathbf{F} \cdot ds,$$

where (as always) $\mathbf{n}$ denotes the unit normal vector at any point on S.

It bears noting that if we have a parametrization φ of our surface S, we can explicitly write this vector $\mathbf{n}$ as $\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} / \left|\left| \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} \right|\right|$.)

We use this theorem in pretty much all of the same places that we use Green's theorem:

- Turning integrals of functions over really awful boundaries into integrals of curls or divergences over surfaces. Often, this process of taking a curl or divergence will make our function 0, or at the least quite trivial.
- If you're integrating something of the form $(\nabla \times f) \cdot n$ over a region, you can of course use the other direction of our proof to consider an integral over the boundary. In practice, this might not come up too often, as it's not always obvious when a given expression is a curl, or the dot product of a curl with a normal vector, or a divergence; so don't look for this unless you're really stuck, or the problem explictly gives you your function in one of these forms.

To illustrate how this goes, we work an example:

Example. If $F(x, y, z) = (-xy^2, x^2y, z)$ and S is the sphere cap $\{(x, y, z) : x^2 + y^2 + z^2 = 25, z \geq 4\}$, find the integral of $(\nabla \times F) \cdot n$ over S.

Solution. As you may have noticed from examples in the notes earlier, sphere caps are sometimes frustrating to work with. So, instead of integrating over this one, let's use Stokes' theorem to instead integrate along its boundary!

[1] A set S is called **bounded** if there is some n such that $||s|| < n$, for all $s \in S$

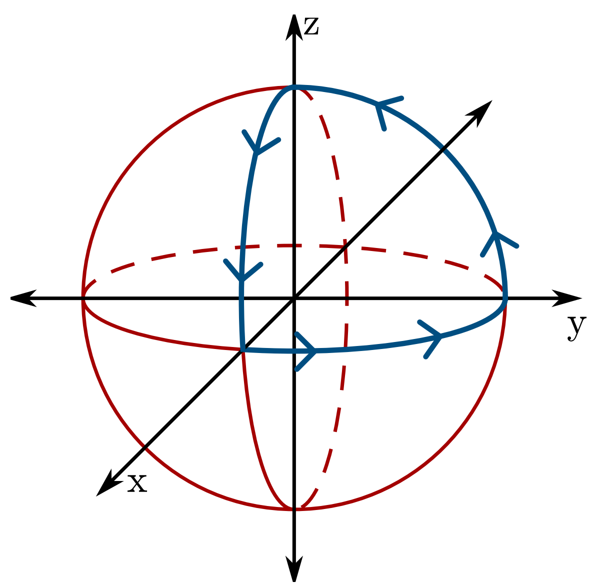
Specifically: the sphere cap above has boundary

$$\partial S = \{(x, y, z) : x^2 + y^2 + z^2 = 25, z = 4\} = \{(x, y, z) : x^2 + y^2 = 3^2, z = 4\},$$

which is traversed in the counterclockwise direction by the curve $\gamma(\theta) = (3\cos(\theta), 3\sin(\theta), 4)$. So, we can use Stokes' theorem to say that

$$\begin{aligned}
\iint_S (\nabla \times F) \cdot n dS &= \int_C F dc \\
&= \int_0^{2\pi} (-xy^2, x^2 y, z)\Big|_{\gamma(\theta)} \cdot \gamma'(\theta) d\theta \\
&= \int_0^{2\pi} (-27\cos(\theta)\sin^2(\theta), 27\cos^2(\theta)\sin(\theta), 4) \cdot (-3\sin(\theta), 3\cos(\theta), 0) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin^3(\theta) + 81\cos^3(\theta)\sin(\theta) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta)(\sin^2(\theta) + \cos^2(\theta)) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta) d\theta \\
&= \int_0^{2\pi} \frac{81\sin(2\theta)}{2} d\theta \\
&= 0.
\end{aligned}$$

Example. Let $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, x, y, z \geq 0\}$ and $C^+ = \partial S$ be the boundary of S traversed in the counterclockwise direction from high above the z-axis, as depicted below:



Let $F(x,y,z) = (x^4, y^4, z^4)$ be a vector field. Calculate $\int_{C^+} F \cdot dc$ directly, then use Stokes's theorem to calculate it with much less effort.

Solution. To calculate this directly, first parametrize C as the three curves $\gamma_1, \gamma_2, \gamma_3$, where

$$\begin{aligned}
\gamma_1(t) &= (\cos(t), \sin(t), 0), \\
\gamma_2(t) &= (0, \cos(t), \sin(t)), \\
\gamma_3(t) &= (\sin(t), 0, \cos(t)),
\end{aligned}$$

and t ranges from 0 to $\pi/2$ for each curve.

Then, we'd have that

$$\begin{aligned}
\int_C F dC &= \sum_{i=1}^{3} \int_0^{\pi/2} (F \circ \gamma_i(t)) \cdot (\gamma'(t)) dt \\
&= \sum_{i=1}^{3} \int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t) dt \\
&= 3 \int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t) dt \\
&= \left(-3 \int_0^{\pi/2} \cos^4(t)\sin(t) dt\right) + \left(3 \int_0^{\pi/2} \sin^4(t)\cos(t) dt\right)
\end{aligned}$$

To evaluate these last two integrals, use the u-substitution $u = \cos(t)$ on the left and $u = \sin(t)$ on the right:

$$\begin{aligned}
\int_C F dC &= \left(3 \int_1^0 u^4 dt\right) + \left(3 \int_0^1 u^4 dt\right) \\
&= \left(-3 \int_0^1 u^4 dt\right) + \left(3 \int_0^1 u^4 dt\right) \\
&= 0.
\end{aligned}$$

Alternately, for a much faster solution, just use Stokes' theorem, which tells us that the integral of F over C is the integral of $(\nabla \times F) \cdot \mathbf{n}$ over S. Then, because

$$\begin{aligned}
\text{curl}(F) = \nabla \times F &= \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right) \\
&= (0-0, 0-0, 0-0) \\
&= (0,0,0),
\end{aligned}$$

we know that $(\nabla \times F) \cdot \mathbf{n}$ is identically 0, and thus that the integral of this quantity over S is also zero.