MATH 1D, FINAL – QUESTIONS

INSTRUCTOR: PADRAIC BARTLETT

Instructions: Choose **four** of the following **nine** questions to complete. (choose *wisely*.) If you do more than that, your top four scores will be graded. As a reminder, the test is open notes/books and collaboration is not allowed; however, you *are* allowed to contact me if you are totally confused about any of the questions, and should not hesitate to do so. Good luck!

0.1. **Calculations.**

Question 0.1. Find the following limits:

$$(A) \qquad \lim_{n\to\infty} \sqrt[n]{n}$$

$$(B) \qquad \lim_{n\to\infty} \sqrt[n]{n^2+n}$$

$$(C) \qquad \lim_{n\to\infty} \frac{a^n - b^n}{a^n + b^n}, \text{ for any } a, b > 0.$$

Question 0.2. Determine whether the following sums converge or diverge, and prove your answer:

$$(A) \qquad \sum_{n=2}^{\infty} \frac{1}{n\log(n)}$$

$$(B) \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Question 0.3. Using Taylor series, evaluate the following infinte sums:

$$(A) \qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$

$$(B) \qquad \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$$

$$(C) \qquad -1 + \log(2) - \frac{(\log(2))^2}{2} + \frac{(\log(2))^3}{3!} - \frac{(\log(2))^4}{4!} + \dots$$

(Hint: look at page 287 in Apostol for all of the Taylor series you'll need to do this question.)

Date: Due Date: Thursday, Feb. 18, at 4 p.m. in the box / 8 p.m. in class.

0.2. Constructions.

Question 0.4. Create a sequence $\{f_n\}$ of discontinuous functions that converge uniformly to a continuous function. Prove that this convergence is indeed uniform.

Question 0.5. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that the integral $\int_0^\infty f(x)dx$ exists, but the integral $\int_0^\infty |f(x)|dx$ does not exist.

(Hint: can you think of a series that is conditionally convergent but not absolutely convergent? How can you turn this series into a function?)

Question 0.6. Find all of the complex numbers z such that

- $z^{12} = 1$, but
- $z^k \neq 1$, for any natural number k less than 12.

What is their product? What is their sum?

(Hint: you can do this directly using the polar expression $re^{i\theta}$ for a complex number.)

0.3. Revisiting Old Problems.

Question 0.7. In week six, we used the three equations

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots,$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots, \text{ and}$$
$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots$$

to show that $e^{iz} = \cos(z) + i\sin(z)$.

Using this, prove that

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Question 0.8. On the third problem of homework 4, we showed that a power series of an even function $f(x) = \sum_{n=0}^\infty a_n x^n$ has $a_n = 0$ whenever n is odd, and that a power series of an odd function $f(x) = \sum_{n=0}^\infty a_n x^n$ has $a_n = 0$ whenever n is even.

Using ideas similar to that proof, take any complex power series $f(z) = \sum_{n=0}^\infty a_n z^n$, such that $f(z) = f(iz)$, and show that $a_n = 0$ whenever n is not a multiple of 4.

Question 0.9. On the first problem of homework 2, we proved that there were a pair of sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers such that

- the sums $\sum_{n=1}^\infty \frac{1}{a_n}$ and $\sum_{n=1}^\infty \frac{1}{b_n}$ both diverged, but
- the sum $\sum_{n=1}^\infty \frac{1}{a_n+b_n}$ converged.

Find another such pair of sequences that satisfy the above conditions, **and** that are strictly increasing.