MATH 1D, WEEK 1 – SEQUENCES

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Abstract. These are the lecture notes from week 1 of Ma1d, the Caltech mathematics course on sequences and series.

1. Class Information

- Class Hours: 8-9 pm, on Tuesdays and Thursdays in 151 Sloan.
- Office Hours: 5-6 pm and 10-11 pm on Wednesday, in 360 Sloan.
- Contact Information: My email address is padraic@caltech.edu, and my office is 360 Sloan.
- Class Website: www.its.caltech.edu/∼padraic
- Homework and Tests: There will be four homework sets, each valued at 10% of your final grade, one midterm valued at 30% of your final grade, and one final valued at 30% of your final grade. Homework sets will generally be about 4 problems long; unless otherwise stated, you are expected to prove that all of your answers are correct. Problems will tend to be short on calculation and long on conceptualization, so if you find yourself looking at a long or "ugly" solution, try again! There is almost certainly an easier way.

  As well, I highly recommend that you look at the problem sets before, say, Wednesday night in office hours – many of the problems in this class will not have an "obvious" solution at first glance, and sleeping on them for a few days will make life much easier for you as students.

  The homework policy is that all textbooks and the Internet is fair game, provided that you don't go looking through online forums for solutions to your questions. Collaboration, as well, is strongly encouraged. However, make sure to write up all of your problems independently, cite any texts that you use if you take parts of solutions from them, and write up everything in your own words (something that everyone should do in any class they are in.) The midterm and final policy is the same, except no collaboration is allowed.
- Homework due date: Thursdays at 4 pm, in the Ma1d class box.

2. Class Overview

This course aims to "fill in" the gaps between the material covered in Ma1a, section 1 and the normal sections of Ma1a; specifically, it aims to cover the concepts of sequences and series as they extend to the real and complex numbers. The following is a rough road map of the topics we will be covering throughout the course:

- **Sequences**:
  - basic definitions
  - concept of convergence
  - the Bolzano-Weierstrass theorem
  - Cauchy sequences, and their connections to convergence
- **Series**:
  - basic definitions
  - various criteria and convergence tests: i.e. the comparison, ratio, and integral tests
  - absolute convergence; its definition and several applications
- **Extensions of the previous concepts beyond the real numbers**:
  - sequences and series of complex numbers; additional convergence tests
  - sequences and series of functions, with attention to the concepts of
    * uniform convergence
    * infinite Taylor series
    * power series, both real and complex

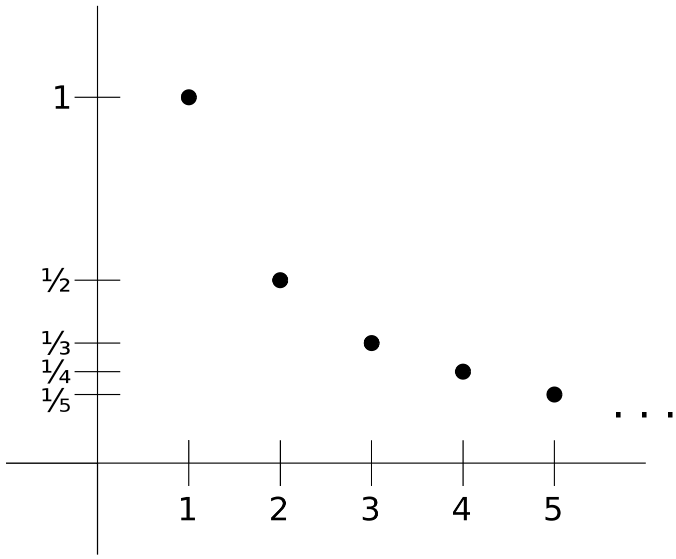
3. Sequences - Basic Definitions

The three lists below are all examples of sequences. The first is the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$, the second is the sequence $\{n^2\}_{n=1}^{\infty}$, and the third is $\{(-1)^n\}_{n=1}^{\infty}$.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots$$
$$1, 4, 9, 16, 25, 36, 49, \ldots$$
$$-1, 1, -1, 1, -1, 1, \ldots$$

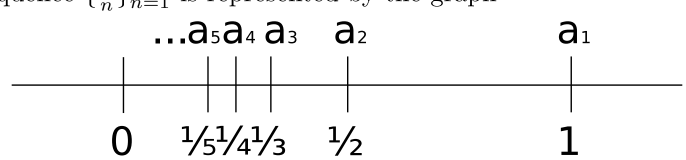
Sequences of real numbers are fairly intuitive mathematical concepts; we often think of them as just "infinite lists of numbers," indexed by the natural numbers $1, 2, 3, 4, \ldots$. We can rigorously define a sequence by calling any labeled collection of numbers $a_1, a_2, a_3, a_4 \ldots$, where there is one a_n for every natural number n, a sequence; more simply, we can say that a sequence of real numbers is any function from $\mathbb{N} \to \mathbb{R}$. Regardless of which definition you choose, however, the idea should be fairly clear; a sequence is simply a infinite list of real numbers. In practice, we will usually denote sequences by the expression $\{a_n\}_{n=1}^{\infty}$, where the a_n stand for the various entries in the sequence.

So: as always, whenever we introduce a new concept, we like to come up with ways of visually understanding it. One way that comes to mind is to graph is as though it was a function from $\mathbb{N}$ to $\mathbb{R}$: in this case, the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is represented by the following graph:



The problem with this view is that it only depicts a tiny amount of the entire sequence – from looking at this picture, we don't have any good idea as to what the "entire" sequence might look like, or where it is concentrated.

There are a number of solutions to this issue that can be created, but one of the simplest is to just graph the sequence as a series of points on a number line. In this way, the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is represented by the graph



The advantage of this picture is that by looking at it, we can get a good feel for where our sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is "going" – i.e. by looking at the graph, we can see intuitively that the terms of this sequence are "heading" to zero.

To make this notion of "heading" to a number more precise, we offer the following definition:

Definition 3.1. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to a number l, and write $\lim_{n\to\infty} a_n = l$, if and only if for every $\epsilon > 0$ there is a natural number N such that for all $n > N$,

$$|a_n - l| < \epsilon.$$

This should remind you of our definitions last quarter for the limits at infinity of a function – they look almost completely the same.

Consequently, several of our theorems from last quarter carry over to this class; we will omit their proofs here, as they are mostly identical to the proofs of these claims for limits we studied earlier:

- **Additivity of sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then so does $\lim_{n\to\infty} a_n + b_n$.
- **Multiplicativity of sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then so does $\lim_{n\to\infty} a_n b_n$.
- **Quotients of sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then so does $\lim_{n\to\infty} \frac{a_n}{b_n}$.
- **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l.
- **Composition of sequences and functions**: if $\lim_{x\to c} f(x) = l$, $\lim_{n\to\infty} a_n = c$, f is defined on every value of a_n, and $a_n \neq c$, then $\lim_{n\to\infty} f(a_n)$ exists and is equal to l. (If f is continuous, we can omit the condition that $a_n \neq c$.)

4. Sequences - Examples

To illustrate this concept of convergence, we calculate a series of examples:

Example 4.1.

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Proof. Pick any $\epsilon > 0$, and choose any natural number N larger than $\frac{1}{\epsilon}$. Then, for every $n > N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \frac{1}{1/\epsilon} = \epsilon.$$

Then, by the definition of convergence, we have that $\lim_{n\to\infty} \frac{1}{n} = 0$, as claimed. □

Example 4.2. Let b be a number in the interval $(0, 1)$. Then

$$\lim_{n\to\infty} b^n = 0.$$

Proof. So: because $b \in (0, 1)$, we know that $\log(b)$ exists and is a negative number. Consequently, we can rewrite

$$b^n = e^{n\cdot\log(b)} = \frac{1}{e^{n\cdot|\log(b)|}}.$$

Then, for any $\epsilon > 0$, pick N to be a natural number such that

$$\log(1/\epsilon) < N \cdot |\log(b)|.$$

Then, we have that

$$\begin{aligned} & e^{\log(1/\epsilon)} < e^{N\cdot|\log(b)|} \\ \Leftrightarrow \quad & 1/\epsilon < e^{N\cdot|\log(b)|} \\ \Leftrightarrow \quad & \epsilon > \frac{1}{e^{N\cdot|\log(b)|}} = b^n = |b^n - 0|. \end{aligned}$$

Again, by the definition of convergence, we have that $\lim_{n\to\infty} b^n = 0$. □

Example 4.3.

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Proof. So: using a similar idea to our earlier example, rewrite

$$\sqrt[n]{n} = (n)^{1/n} = e^{\log(n)/n}.$$

Then, look at the sequence $\{\log(n)/n\}_{n=1}^{\infty}$. We claim that this sequence converges to zero. To see this: look at the function $\frac{\log(x)}{x}$.

By L'Hôpital's rule, we know that

$$\lim_{x\to\infty} \frac{\log(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

But what does this actually mean? Well, by the definition of a limit at infinity, this means that for any $\epsilon > 0$ there is some N such that for all $x > N$, $|\log(x)/x - 0| < \epsilon$ – so, in particular, for any natural number $n > N$ we have that $|\log(n)/n - 0| < \epsilon$. But that means that

$$\lim_{n\to\infty} \frac{\log(n)}{n} = 0,$$

as we claimed.

Then, because

- $\lim_{x\to 0} e^x = 1$,
- $\lim_{n\to\infty} \frac{\log(n)}{n} = 0$, and
- $e^{\log(n)/n}$ is defined for every n,

we have (by composing our sequence $\log(n)/n$ with the sequence e^x) that

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\log(n)/n} = 1.$$

□

So, in the above proof, we did one interesting thing that it turns out holds in complete generality – we used a continuous function to prove something about a discrete function. Explicitly, we used the observation that

$$\lim_{x\to\infty} \frac{\log(x)}{x} = 0$$

to prove that

$$\lim_{n\to\infty} \frac{\log(n)}{n} = 0$$

It turns out that this holds in general! I.e. we have the following proposition:

Proposition 4.4. *If* $\{a_n\}_{n=1}^{\infty}$ *is a sequence,* f *is a function such that* $f(n) = a_n, \forall n$, *and* $\lim_{x\to\infty} f(x)$ *exists, then*

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

The proof of this statement is pretty much exactly what we did in our last example; there is nothing special about $\log(x)/x$.

5. Sequences - Main Theorems

So: the above has hopefully illustrated what a sequence is, and what it means for a sequence to converge. However, we don't really have much machinery to study sequences properly yet – in all of our examples, we worked straight from the definitions or just used analogues of theorems from the continuous case of limits. While this sufficed for everything we did above, there are in fact a fairly rich set of theorems dealing with the study of sequences: however, before we can present these theorems, we must begin with a few basic definitions.

Definition 5.1. We call a sequence $\{a_n\}_{n=1}^{\infty}$

- **increasing** if $a_n < a_{n+1}$, for every n,
- **nondecreasing** if $a_n \leq a_{n+1}$, for every n, and
- **bounded above** if there is some number M such that $a_n \leq M$, for every n.

Similarly, we say that a sequence $\{a_n\}_{n=1}^{\infty}$ is

- **decreasing** if $a_n > a_{n+1}$, for every n,
- **nonincreasing** if $a_n \geq a_{n+1}$, for every n, and
- **bounded below** if there is some number M such that $a_n \geq M$, for every n.

(These definitions are exactly the same as they were for functions, and thus should hopefully look familiar.)

Definition 5.2. A **subsequence** of the sequence $\{a_n\}_{n=1}^{\infty}$ is a sequence of the form

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots$$

where the n_i are an infinite sequence of natural numbers such that

$$n_1 < n_2 < n_3 < n_4 < n_5 \ldots$$

Essentially, a subsequence is just what you get by taking a sequence and "skipping over" some of its entries. For example, the sequence

$$(\ddagger) \qquad 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

has both

$$0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots \; 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$$

as subsequences, because choosing all of the even entries in $(\ddagger)$ gives you $0, 0, 0, 0 \ldots$ whereas choosing the odd entries gives you the sequence $1, 1, 1, 1 \ldots$.

Similarly, the sequence

$$(\star) \qquad 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0, 5, 0, 0, \ldots$$

has the sequence

$$1, 2, 3, 4, 5, 6, \ldots$$

as a subsequence realized by only choosing every third entry in $(\star)$.

These concepts presented, we can now move on to the subjects of this lecture:

Theorem 5.3. *If $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges.*

Proof. So: let A be the set consisting of all of the numbers a_n in our sequence. A is bounded above, because $\{a_n\}_{n=1}^{\infty}$ is; so A has a **least upper bound**! Call it α. (A least upper bound α, for those of you who don't remember, is an upper bound for A such that every other upper bound β of A is bigger than α.)

So: we claim that

$$\lim_{n\to\infty} a_n = \alpha.$$

To see this: choose any $\epsilon > 0$. Because α is a least upper bound, we know that no number smaller than α can be an upper bound – so, in specific, $\alpha - \epsilon$ is not an upper bound. But what does this mean? Just that there is some $a_N \in A$ such that

$$a_n > \alpha - \epsilon \Leftrightarrow \epsilon > \alpha - a_N.$$

But because the a_n are nondecreasing and bounded above by α, we know that actually

$$\epsilon > |\alpha - a_n|, \forall n > N.$$

So $\lim_{n\to\infty} a_n$ converges (specifically, to α,) just as we claimed. □

It bears noting that the same result holds if we instead stipulate that $\{a_n\}_{n=1}^{\infty}$ is nonincreasing and bounded below, by just doing the same proof with greatest lower bounds instead of least upper bounds.

Before moving onto our second theorem, we pause here to note an application of this theorem, which allows us to determine if a sequence converges **without even knowing what it converges to**:

Example 5.4. Let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of numbers $\beta_n \in [2, \infty)$, and let

$$a_n = \sum_{k=1}^{n} \frac{1}{\beta_k^k}.$$

We claim that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

Proof. To see this, simply note that

- because
$$a_{n+1} - a_n = \sum_{k=1}^{n+1} \frac{1}{\beta_k^k} - \sum_{k=1}^{n} \frac{1}{\beta_k^k} = \frac{1}{\beta_{n+1}^{n+1}} > 0,$$
this sequence is increasing.
- Because
$$a_n = \sum_{k=1}^{n} \frac{1}{\beta_k^k} < \sum_{k=1}^{n} \frac{1}{2^k} = \frac{2^n - 1}{2^n} < 1,$$
we know that this sequence is bounded above (where the identity $\sum_{k=1}^{n} \frac{1}{2^k} = \frac{2^n-1}{2^n}$ is an inductive identity we proved in the first quarter of this course.)

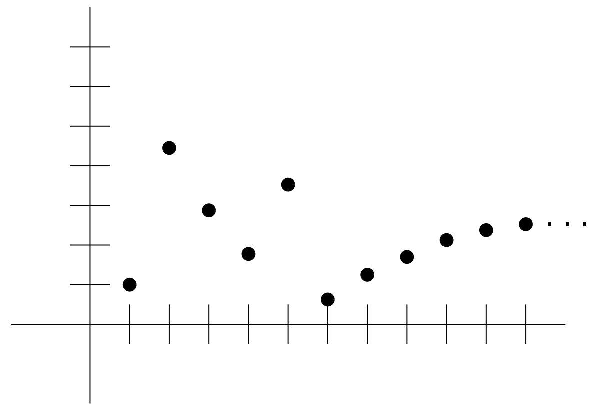
As a result, we can simply apply our earlier theorem to conclude that the sequence $\{a_n\}_{n=1}^{\infty}$ converges, even though we have no idea what it converges to! □

So: the above theorem allows us to show that sequences converge, even though we may not know what they converge to. The next theorem is similar in flavor to this first one; it too guarantees the existence of some object, without necessarily telling us what it is.

Theorem 5.5. *Any sequence $\{a_n\}_{n=1}^{\infty}$ has a subsequence that is either nondecreasing or nonincreasing.*

Proof. This proof relies entirely on one crucial concept:

Definition 5.6. Call a number $n \in \mathbb{N}$ a **peak point** of a sequence $\{a_n\}_{n=1}^{\infty}$ if $a_m < a_n$, for all $m > n$. (In the sequence graphed below, for example, 2 and 5 are peak points.)



So, there are trivially two possible cases we have to consider: either

- there are infinitely many peak points, or
- there are only finitely many peak points.

If we are in the first case: let $n_1 < n_2 < n_3 < n_4 \ldots$ be an ordered infinite list of the peak points. Then, by definition, $a_{n_k} > a_{n_{k+1}}$, for every k, because these are all peak points! So this is a decreasing sequence.

So it suffices to consider the second case, where there are only finitely many peak points. Let N be the collection of all of the peak points. This is a finite set; so there is some natural number n_1 such that $n_1 > m, \forall m \in N$. But this means specifically that n_1 is not a peak point (as it's bigger than every peak point) – so, by definition, there has to be some value $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. As well, n_2 is also not a peak point, because it is bigger than n_1 and thus all of N; so there must be some n_3 such that $n_3 > n_2$ and $a_{n_3} \geq a_{n_2}$. Continuing inductively, we get an infinte sequence of values n_i such that $a_{n_1} \leq a_{n_2} \leq a_{n_3} \ldots$ – i.e. a nondecreasing sequence! So we are done. □

While this theorem is often used to study sequences by itself, its most celebrated application is to the following corollary:

Corollary 5.7. *(Bolzano-Weierstrass Theorem) Any bounded sequence must contain a convergent subsequence.*

Proof. By the theorem above, we know that every bounded sequence has to have either a nondecreasing or a nonincreasing bounded subsequence; but any such subsequence must converge to some value by our earlier theorem! □