MATH 1D, WEEK 2 – CAUCHY SEQUENCES, LIMITS SUPERIOR AND INFERIOR, AND SERIES

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Abstract. These are the lecture notes from week 2 of Ma1d, the Caltech mathematics course on sequences and series.

1. Limits Superior and Inferior

So: most of the definitions and theorems we've developed so far for sequences are centered around the concept of **convergence** – we have lots of ways of talking about when things converge, where they converge to, and under what conditions they will be forced to converge.

However, when we're confronted with a divergent sequence, it is sometimes useful to be able to say more about it than just "it doesn't converge!" For example, the sequences

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots$$
$$\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{6}{7}, \dots$$

both diverge, and yet both exhibit very clear behaviors at infinity – specifically, both sequences seem to "tend" to both 0 and 1 at infinity. The following definition helps us offer a canonical way of talking about such limiting behaviors at infinity, even when looking at such divergent sequences:

Definition 1.1. For a sequence $\{a_n\}$, set $x_n = \sup\{a_m : m \geq n\}$. We then define the **limit superior** of a_n as

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} x_n.$$

Similarly, if we set $y_n = \inf\{a_m : m \geq n\}$, we can then define the **limit inferior** of a_n as

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} y_n.$$

Example 1.2. If $\{a_n\}$ is the sequence given by

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots$$

then for any n we have that

$$\sup\{a_m : m \geq n\} = 1,$$
$$\inf\{a_m : m \geq n\} = 0,$$

and thus that

$$\limsup_{n\to\infty} a_n = 1, \liminf_{n\to\infty} a_n = 0.$$

Example 1.3. If $\{a_n\}$ is the sequence given by

$$\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{6}{7}, \ldots$$

then for any n we have that

$$1 \geq \sup\{a_m : m \geq n\} \geq \frac{k-1}{k}, \text{ for any sufficiently large } k,$$

because every other entry in the sequence $\{a_n\}$ is of the form $\frac{k-1}{k}$, where k increases with n. Consequently, we have that $\limsup_{n\to\infty} a_n = 1$! Similarly, for any n we have that

$$0 \leq \inf\{a_m : m \geq n\} < \frac{1}{k}, \text{ for any sufficiently large } k,$$

and thus that $\liminf_{n\to\infty} a_n = 0$.

So: much as with normal limits, we have a number of useful conditions as to when the limits superior and inferior exist. We state and prove these all below:

Theorem 1.4. *If $\{a_n\}$ is a bounded sequence, then $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ both exist.*

Proof. Note that for every n, we have that

$$x_n := \sup\{a_m : m \geq n\} \geq x_{n+1} := \sup\{a_m : m \geq n+1\},$$

and thus that the sequence of supremums $\{x_n\}$ is nonincreasing. As well, because $\{a_n\}$ is a bounded sequence, the $x_n's$ are also bounded; combining these two results then tells us that $\{x_n\}$ is a convergent sequence – i.e. that $\limsup_{n\to\infty} a_n$ exists!

Similarly, by noting that

$$y_n := \inf\{a_m : m \geq n\} \leq x_{n+1} := \inf\{a_m : m \geq n+1\},$$

we have that the y_n's form a nondecreasing bounded sequence, and thus also converge - i.e that $\liminf_{n\to\infty} a_n$ exists as well. □

Theorem 1.5. $\lim_{n\to\infty} a_n$ *exists if and only if* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof. Suppose first that $\lim_{n\to\infty} a_n$ exists, and furthermore is equal to some l. Then, by definition, we have that for any $\epsilon > 0$, there is some N such that for every $n > N$, a_n is within ϵ of l.

But this means that (in particular) the biggest of all of the a_n's with $n > N$ has to be within ϵ of l – i.e that $\sup\{a_m : m \geq n\}$ must be within ϵ of l.

But this is literally the definition of the statement

$$\lim_{n\to\infty} \sup\{a_m : m \geq n\} = l;$$

which, in other words, is just the statement

$$\limsup_{n\to\infty} a_n = l.$$

So the limit superior exists. A completely identical argument (just replace the inf's with sup's above) shows that the limit inferior exists as well; so we have shown one direction of our "if and only if" statement.

To prove the other direction, assume now that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. What happens then?

Well – we have, for every a_n, the relation

$$\inf\{a_m : m \geq n\} \leq a_n \leq \sup\{a_m : m \geq n\}.$$

Thus, by the squeeze theorem for sequences, we have that $\lim_{n\to\infty}$ exists and is equal to both the $\limsup$ and $\liminf$. □

2. Cauchy Sequences

Definition 2.1. We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m, n \geq N$

$$|a_m - a_n| < \epsilon.$$

You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

So: a priori, this definition looks to be somewhat different than our earlier definition of convergence – in particular, the definition of Cauchy makes no reference to any sort of "limit point," whereas the notion of a sequence having a limit point is key to the idea of convergence. But it seems plausible that these definitions might be equivalent – after all, both conditions are basically ways of saying that a sequence "stops moving at infinity." This actually turns out to be the case, as the next theorem states:

Theorem 2.2. *A sequence is Cauchy if and only if it converges.*

Proof. Suppose that $\{a_n\}$ is a sequence that converges to some value l; then, by definition, we have that

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t } \forall n > N, |a_n - l| < \epsilon.$$

So, in particular, if we pick any m, n both larger than N, we have that both $|a_n - l| < \epsilon$ and $|a_m - l| < \epsilon$. Adding these two results together tells us that

$$\begin{aligned}
&|a_m - l| + |a_n - l| < 2\epsilon \\
\Rightarrow &|a_m - l| + |l - a_n| < 2\epsilon \\
\Rightarrow &|a_m - l + l - a_n| < 2\epsilon \\
\Rightarrow &|a_m - a_n| < 2\epsilon,
\end{aligned}$$

(where the above implications follow from the triangle inequality and simple algebraic manipulations.)

But this is precisely the condition that $\{a_n\}$ is Cauchy! So we have that all convergent sequences are Cauchy.

To see the other direction – suppose that $\{b_n\}$ is a Cauchy sequence. We seek now to show that this sequence converges.

To do this: notice first that by definition, we have that for every $\epsilon > 0$ there is a natural number N such that for every $m, n \geq N$

$$|b_m - b_n| < \epsilon.$$

In particular, if we let $\epsilon = 1$ and $n = N$, we have that

$$\forall m \geq N, |b_m - b_N| < 1 \Rightarrow \forall m \geq N, |b_m| < 1 + |b_N|;$$

in other words, that the values of b_n are bounded, for all $n \geq N$. However, because the set $\{b_1, b_2, \ldots b_N\}$ is a finite set, we know that it too is bounded, say by some constant C. By taking the larger of these two bounds, we have that in fact the entire sequence $\{b_n\}$ is bounded – thus, by our earlier theorems, we know that it has a convergent subsequence! Denote this convergent subsequence by $\{b_{n_k}\}$, and call the number that it converges to l.

We claim that (in fact) the entire sequence $\{b_n\}$ converges to l. To see this, we just have to note the following two things:

(1) because $\lim_{k\to\infty} b_{n_k} = l$, we know that

$$\forall \epsilon > 0, \exists N_1 \in \mathbb{N} \text{ s.t } \forall n_k > N_1, |b_{n_k} - l| < \epsilon.$$

(2) because the sequence $\{b_n\}$ is Cauchy, we know that

$$\forall \epsilon > 0, \exists N_2 \in \mathbb{N} \text{ s.t } \forall n, m > N_2, |b_m - b_n| < \epsilon.$$

So, in particular, if $N = \max\{N_1, N_2\}$, for every $n_k, m > N$ we have that $|b_{n_k} - l| < \epsilon$ and $|b_m - b_{n_k}| < \epsilon$. Adding these two equations together gives us that

$$\begin{aligned}
&|b_{n_k} - l| + |b_m - b_{n_k}| < 2\epsilon \\
\Rightarrow &|b_{n_k} - l + b_m - b_{n_k}| < 2\epsilon \\
\Rightarrow &|b_m - l| < 2\epsilon,
\end{aligned}$$

(where the above implications follow from the triangle inequality and simple algebraic manipulations.)

But this is the definition of convergence! So we have that the sequence $\{b_n\}$ converges as claimed. $\square$

The following example illustrates why the Cauchy criterion is sometimes useful, as it can be much easier to show that certain things are Cauchy than to show that they converge – in particular, if we don't know where a sequence converges to, it can be **much** easier to show that it's Cauchy than to show it converges.

Example 2.3. Let

$$a_n := \sum_{k=1}^{n} \frac{1}{k^2}.$$

Does a_n converge?

Proof. So: we claim that $\{a_n\}$ is in fact Cauchy. To see this: simply notice that for any $m, n > N$ where $m > n$

$$\begin{aligned} 0 \leq a_m - a_n &= \sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \\ &= \sum_{k=n+1}^{m} \frac{1}{k^2} \\ &< \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\ &= \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} \\ &= \sum_{k=n+1}^{m} \frac{1}{k-1} - \sum_{k=n+1}^{m} \frac{1}{k} \\ &= \sum_{k=n}^{m-1} \frac{1}{k} - \sum_{k=n+1}^{m} \frac{1}{k} \\ &= \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n} + \frac{1}{m} \\ &< \frac{2}{N}. \end{aligned}$$

Thus, if $N > \frac{2}{\epsilon}$, we have that for any $m, n > N$,

$$|a_m - a_n| < \frac{2}{2/\epsilon} = \epsilon.$$

But this just means that our sequence is Cauchy! So, by the above theorem, it converges. □

3. Series and Sums

So: with the above example in mind, we now turn from general sequences to the study of series and infinite sums. We define the concept of an "infinte sum" below:

Definition 3.1. A sequence is called **summable** if the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums

$$s_n := a_1 + \ldots a_n$$

converges. If it does, we then call the limit of this sequence the **sum** of the a_n, and denote this quantity by writing

$$\sum_{n=1}^{\infty} a_n.$$

We first illustrate this definition by a pair of examples:

Example 3.2. For any $r \in (-1,1)$, does the sum

$$\sum_{n=0}^{\infty} r^n$$

converge? If so, what does it converge to?

Proof. So: this can be done by simply performing the following algebraic trick: if we denote the partial sums $r^0 + r^1 + r^2 + \dots r^n$ by the symbols s_n, we have that

$$\begin{aligned} s_n &= 1 + r^1 r^2 + \dots + r^n \\ r \cdot s_n &= r^1 + r^2 + \dots r^{n+1} \\ \Rightarrow s_n - r s_n &= 1 - r^{n+1} \\ \Rightarrow s_n(1-r) &= 1 - r^{n+1} \\ \Rightarrow s_n &= \frac{1 - r^{n+1}}{1-r} \\ \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1-r} &= \frac{1}{1-r}. \end{aligned}$$

But the limit of these partial sums is, by definition, the infinite sum of the r^n's! So this sum converges – specifically, it converges to $\frac{1}{1-r}$. □

Example 3.3. Does the sum

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converge?

Proof. This sum, in fact, does not converge! To see this, simply note that if s_n denotes the partial sums $1 + \frac{1}{2} + \dots + \frac{1}{n}$, we have that

$$s_n = \sum_{k=1}^{n} 1 \cdot \frac{1}{k} \geq \int_1^n \frac{1}{x} dx = \log(n),$$

and thus that

$$\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} \log(n) = \infty,$$

and so (in particular) does not converge. □

So: because series are a special type of sequences, it stands to reason that all of our original theorems on sequences should have a nice interpretation in the language of infinte sums. We describe two such quick results here:

Lemma 3.4. *(Cauchy criterion) A sequence $\{a_n\}$ is summable if and only if*

$$\lim_{m,n\to\infty} |a_n + a_{n+1} + \dots + a_m| = 0$$

Proof. Just place the sequence $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ into the definition of a Cauchy sequence, and apply our earlier theorem that says that sequences are Cauchy if and only if they converge. □

Theorem 3.5. *(Vanishing criterion) If $\{a_n\}$ is summable, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Let $m = n+1$ in the limit $\lim_{m,n\to\infty} |a_n + a_{n+1} + \ldots + a_m| = 0$, which (by the above) is equivalent to summability. $\square$

Theorem 3.6. *(Boundedness criterion:) Suppose that $\{a_n\}$ is a sequence of non-negative numbers such that the collection of all partial sums $\sum_{k=1}^n a_n$ is bounded. Then $\{a_n\}$ is summable.*

Proof. So: because the a_n are all nonnegative, we have that the sequence $\{\sum_{k=1}^n a_n\}$ is nondecreasing. By assumption, it's also bounded; so it must converge! $\square$

These results, admittedly, are pretty trivial, and aren't really "new" ideas. The following theorem, however, is a tool that's markedly more powerful than anything we have in the world of just general sequences:

Theorem 3.7. *(First Comparison Test:) Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences such that$0 \le a_n \le b_n$, for all n. Then $\{a_n\}$ is summable if $\{b_n\}$ is summable.*

Proof. So: again, because the a_n are positive, the sequence of partial sums $\{\sum_{k=1}^n a_n\}$ is nondecreasing. As well, because

$$\sum_{k=1}^{n} a_n \le \sum_{k=1}^{n} b_n \le \sum_{k=1}^{\infty} b_n = \text{ some fixed constant } l < \infty,$$

the sequence of partial sums is bounded; so it converges! $\square$

This test can be thought of as a weirder form of the squeeze theorem – in that to show that a sequence converges, we merely need to show that it is termwise positive and less than another sequence that converges, **not** that it is pinched by a pair of sequences that converge to the same place! In fact, most of the time when we apply the comparison test, we will be bounding $\sum^\infty a_n$ above by a sum that converges to a value much large than $\sum^\infty a_n$. Some examples follow below:

Example 3.8. Let

$$a_n = \frac{2}{2^n - \sin(n)}.$$

Is a_n summable?

Proof. Well,

$$a_n = \frac{2}{2^n - \sin(n)} \le \frac{2}{2^n - 1} \le \frac{2}{2^{n-1}} = 4 \cdot \frac{1}{2^n},$$

and $\sum_{n=1}^\infty 4 \cdot \frac{1}{2^n} = 4 \cdot \sum_{n=1}^\infty \frac{1}{2^n} = 4 \cdot 1 = 4$ converges; thus, by the first comparison test, so does $\sum^\infty a_n$. $\square$

Example 3.9. Let

$$a_n = \frac{3n^3}{4n^4 - 1}$$

Is a_n summable?

Proof. Well,

$$a_n = \frac{3n^3}{4n^4 - 1} \ge \frac{3n^3}{4n^4} = \frac{3}{4} \cdot \frac{1}{n},$$

and $\sum_{n=1}^\infty \frac{3}{4} \cdot \frac{1}{n} = \frac{3}{4} \cdot \sum_{n=1}^\infty \frac{1}{n}$ diverges; as a result, so does $\sum^\infty a_n$. $\square$

As suggested by its name, there is another comparison test beyond the first comparison test, which we describe below:

Theorem 3.10. *(Second Comparison Test:) Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are a pair of positive sequences such that* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and is equal to some ratio* $c \neq 0$. *Then* $\{a_n\}$ *is summable if and only if* $\{b_n\}$ *is summable.*

Proof. Note first that because $c \neq 0$, we have that both the limits $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{c}$ exist. Thus, without any loss of generality, if we want to show that $\{a_n\}$ is summable if and only if $\{b_n\}$ is summable, we can just assume that that $\{b_n\}$ is summable, and show that $\{a_n\}$ is consequently summable (as by our argument above, they're completely interchangeable.)

So: because $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, we have (by definition) that there is some N such that for all $n > N$,

$$\frac{a_n}{b_n} \leq 2c$$
$$\Rightarrow a_n \leq 2cb_n$$

So: because the a_n's are positive and the sequence $\sum^\infty 2cb_n = 2c\sum^\infty b_n$ converges, the first comparison test tells us that the sum $\sum^\infty a_n$ must also converge! This completes our proof. $\square$

We conclude with a quick example of the use of this theorem:

Example 3.11. Let $a_n = \frac{(n+1)^3}{4^n}$. Is $\{a_n\}$ summable?

Proof. Well, if $b_n = \frac{(n)^3}{4^n}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{4^n}}{\frac{(n)^3}{4^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 = 1;$$

so, by the second comparison test, $\{a_n\}$ is summable if and only if $\{b_n\}$ is. But

$$\frac{(n)^3}{4^n} \leq \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n,$$

which we know to be summable; so by the first comparison test, $\{b_n\}$ is summable! Consequently, $\{a_n\}$ is summable as well. $\square$