Math 1d	Instructor: Padraic Bartlett

Homework 1

Week 2 — *Caltech - Winter 2012*

Instructions: Choose **three** questions out of the following **six** to complete. If you attempt more than three questions, your three highest scores will be recorded as your grade. Also, write me/Daiqi me an email if you get stuck!

Resources allowed: Wikipedia, Mathematica, your notes, the online class notes, textbooks, and your classmates. If you've completed a problem via Mathematica or writing a program, attach your code to receive credit.

Standard Exercises:

1. Show that the claimed limits of the following sequences are true:

   (a)
   $$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{1/n} = 1.$$

   (b)
   $$\lim_{n\to\infty} (x^n + y^n)^{1/n} = y,$$

   where x, y are a pair of positive real numbers such that $x < y$. (Hint: squeeze!)

2. (a) Create a sequence $\{a_n\}_{n=1}^{\infty}$ that doesn't contain **any** convergent subsequences. Justify your answer.

   (b) Characterize all of the subsequences of the sequence
   $$1, 0, 1, 0, 1, 0, 1, 0 \ldots$$
   that converge. (Again, justify your answer.)

3. (a) Show that if x is a real number such that $0 < x < 2$, then $0 < x < \sqrt{2x} < 2$.

   (b) Use (a) to show that the sequence
   $$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$
   is monotone and bounded, and therefore converges.

   (c) What does it converge to?

More Interesting Problems:

4. The Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ is defined as follows:

$$f_0 = 0, f_1 = 1, f_{n+2} = f_n + f_{n+1}.$$

   (a) Write the first 10 terms of the Fibonacci sequence.
   (b) Using induction, prove that

$$f_n = \frac{\varphi^n - (-1/\varphi)^n}{\sqrt{5}},$$

   where $\varphi = \frac{1+\sqrt{5}}{2}$ denotes the **golden ratio**. (Hint: notice that $\varphi^2 = \varphi + 1$, and similarly that $(-1/\varphi)^2 = (-1/\varphi) + 1$, and use this in your induction step.)
   (c) Using part (c), prove that

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi.$$

5. For any positive integer k, the k-**hailstone** sequence $\{h_n\}_{n=0}^{\infty}$ is defined as follows:
   - Define $h_0 = k$.
   - If h_n is odd, define $h_{n+1} = 3h_n + 1$.
   - If h_n is even, define $h_{n+1} = \frac{h_n}{2}$.

   For example, the following sequence is the 13-hailstone sequence:

$$13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots$$

   The **Collatz conjecture**, an open problem in mathematics, is the claim that no matter what number you start with, this sequence will eventually reach 1.

   (a) Write a computer program to verify that the Collatz conjecture is true for all positive natural numbers less than 1000.
   (b) In our example sequence where we started at 13, we got to 1 after 9 steps, i.e. at the 9th entry of our sequence. Using your program, determine which number less than 1000 takes the most steps to get to 1. How many steps does it take?

6. The following is the first five entries in the **look-and-say** sequence:

$$\begin{aligned} &1, \\ &11, \\ &21, \\ &1211, \\ &111221, \ldots \end{aligned}$$

   To generate the next entry of the "look-and-say" sequence from the most recent entry, simply read the last entry aloud, counting the number of digits in groups of that digit. For example, starting from 111221, we would read

- 111 is read off as“three ones,” i.e. 31.
- 22 is read off as “two twos,” i.e. 22.
- 1 is read off as “one one,” i.e. 11.

So the next entry in our sequence is 312211.

(a) Write the next three entries of the look-and-say sequence.

(b) Prove that no element of the look-and-say sequence will ever contain a 4.