Math 1d	Instructor: Padraic Bartlett

Homework 6

Week 7	*Caltech - Winter 2012*

Instructions: Choose **three** questions out of the following **six** to complete. If you attempt more than three questions, your three highest scores will be recorded as your grade. Also, write me/Daiqi an email if you get stuck!

Resources allowed: Wikipedia, Mathematica, your notes, the online class notes, textbooks, and your classmates. If you've completed a problem via Mathematica or writing a program, attach your code to receive credit.

Standard Exercises:

1. (a) Check that

$$\frac{1}{1+z^2} = \frac{1}{2i} \cdot \left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

   (b) Using the expansion at the right along with the product/quotient rules (i.e. don't use the definition of the derivative!), find

$$\frac{d^n}{dz^n}\left(\frac{1}{1+z^2}\right),$$

   for any $n \in \mathbb{N}$.

   (c) Using the result in (b), tell me what

$$\frac{d^n}{dx^n}\left(\arctan(x)\right)\Big|_{x=0}$$

   is, for any $n \in \mathbb{N}$.

2. Pick any nonzero complex number $z = x + iy$, where $x, y \in \mathbb{R}$. Find a pair of values $a, b \in \mathbb{R}$ such that if $w = a + bi$, we have

$$z \cdot w = 1.$$

3. Consider the following three series:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=1}^{\infty} z^n.$$

   (a) Show that the radius of convergence of each of these power series is 1.

   (b) Show that the first series converges everywhere on the unit circle (i.e. the subset of $\mathbb{C}$ with norm 1: i.e. the subset of $\mathbb{C}$ given by $\{e^{i\theta} : \theta \in [0, 2\pi)\}$.) Show that the second series diverges at at least one point on the unit circle, and converges at at least one other point on the unit circle. Show that the third series diverges everywhere on the unit circle.

More Interesting Problems:

4. Let z_1, z_2, z_3 be three elements of $\mathbb{C}$ such that

   - $z_1 + z_2 + z_3 = 0$, and
   - $|z_1| = |z_2| = |z_3|$.

   Show that these three points, as graphed in $\mathbb{C}$, form the vertices of an equilateral triangle. (Hint: again, use polar coordinates: i.e. write $z_k = r_k \cdot e^{i\theta_k}$. What do you know about the r_i's?)

5. A n^{th}**-root of unity** is a complex number $\omega \in \mathbb{C}$ such that $\omega^n = 1$. For example, -1 and 1 are the only two 2^{nd}-roots of unity, because they are the only two complex numbers ω such that $\omega^2 = 1$.

   (a) Find all of the third roots of unity. (Hint: use polar coördinates. In other words, think of elements of $\mathbb{C}$ as numbers of the form $re^{i\theta}$.)

   (b) Let ω be a n^{th}-root of unity not equal to 1. Show that

   $$\sum_{k=0}^{n-1} \omega^k = 0.$$

   (Hint: Draw all of these powers of ω on the complex plane for some small values of n, like 3 or 4. Graphically, why is this sum 0?)

6. The complex numbers $\mathbb{C}$ were formed by taking the real numbers $\mathbb{R}$ and adding the symbol i, where we thought of i as $\sqrt{-1}$.

   Similarly, we can define the **Gaussian integers** as the set

   $$\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\},$$

   formed by taking the integers and adding the symbol i.

   (a) In the integers, we know that a number is even if and only if it's a multiple of 2. Show that in the Gaussian integers, the square of the norm[1] $|\alpha|^2$ of a Gaussian integer α is even if and only if it is a multiple of $1 + i$: i.e.

   $$2 \Big| |\alpha| \Leftrightarrow \alpha = (1 + i)(c + di),$$

   for some $c, d \in \mathbb{Z}$. (Hint: Feel free to use the following fact without proof: for any two complex numbers α, β, we have $|\alpha \cdot \beta| = |\alpha| \cdot |\beta|$.)

   (b) Find the Gaussian integer $\beta = c + di$ such that $(1 + i)(c + di) = 2$.

   (c) Show that if α, β are a pair of Gaussian integers such that $\alpha \cdot \beta = 1 + i$, one of α, β is equal to one of $\pm 1, \pm i$.

   In this sense, $1 + i$ is a "prime" in the Gaussian integers, in much the same sense that 2 is a prime in the normal integers.

[1] Recall from class: the norm $|z|$ of a complex number $z = x + iy$, $x, y \in \mathbb{R}$ is just the distance from z to the origin: i.e. $\sqrt{x^2 + y^2}$. Therefore, the square of the norm is just $x^2 + y^2$. Notice that this is always a positive integer whenever $x + iy$ is a Gaussian integer.