| Math 1d | Instructor: Padraic Bartlett |
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# Lecture 4: Power Series and Fourier Series

*Week 5* — *Caltech - Winter 2012*

## 1 Random Questions

**Question 1.1.** *A* ***Latin square*** *is a $n \times n$ array filled with the symbols $\{1, \ldots n\}$, so that no symbol is repeated in any row or column.*

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 1 |
| 3 | 4 | 1 | 2 |
| 4 | 1 | 2 | 3 |

*A* ***partial Latin square*** *is a Latin square where we also allow some cells to be blank:*

| | | | |
|---|---|---|---|
| 1 | 4 | | |
| | | 4 | 1 |
| | | 1 | 4 |
| 4 | 1 | | |

*Some partial Latin squares can be completed:*

| | | | |
|---|---|---|---|
| 1 | 4 | | |
| | | 4 | 1 |
| | | 1 | 4 |
| 4 | 1 | | |

$\mapsto$

| | | | |
|---|---|---|---|
| 1 | 4 | 2 | 3 |
| 3 | 2 | 4 | 1 |
| 2 | 3 | 1 | 4 |
| 4 | 1 | 3 | 2 |

*Others cannot be completed:*

| | | | |
|---|---|---|---|
| 1 | | | |
| | 1 | | |
| | | 1 | |
| | | | 2 |

$\mapsto ?$

*Show that if $P$ is a partial latin square in which at most $n/4$ of the cells corresponding to any row, column, or symbol are nonblank, then $P$ can be completed to a latin square.*

*(The current best published bound is $n/10^7$; over the last year, I reduced this to $n/9000$, but my techniques probably aren't going to improve this by much. Novel ideas are needed to probably improve this bound, which is where you come in!)*

**Question 1.2.** *(Lonely runner conjecture): Take a circle of circumference 1. Choose $m$ distinct speeds $(a_1, \ldots a_m$, and to each of these speeds associate a "runner" (i.e. a point) moving at that speed around our circle.*

*A runner $r$ is called* ***lonely*** *at time $t$ if at that time, every other runner is at least distance $\frac{1}{m}$ away from this runner $r$. Show that for any collection of runners and distinct speeds, there is always a time $t$ at which some runner is lonely.*

*(This problem is open for $m \geq 8$.)*

# 2 Power Series

## 2.1 Power Series: Definitions and Tools

The motivation for **power series**, roughly speaking, is the observation that polynomials are really *quite nice*. Specifically, if I give you a polynomial, you can

- differentiate and take integrals easily,
- add and multiply polynomials together and easily express the result as another polynomial,
- find its roots,

and do most anything else that you'd ever want to do to a function! One of the only downsides to polynomials, in fact, is that there are functions that **aren't** polynomials! In specific, the very useful functions

$$\sin(x), \cos(x), \ln(x), e^x, \frac{1}{x}$$

are all not polynomials, and yet are remarkably useful/frequently occuring objects.

So: it would be nice if we could have some way of "generalizing" the idea of polynomials, so that we could describe functions like the above in some sort of polynomial-ish way – possibly, say, as polynomials of "infinite degree?" How can we do that?

The answer, as you may have guessed, is via **power series**:

**Definition 2.1.** A **power series** $P(x)$ centered at $x_0$ is just a sequence $\{a_n\}_{n=1}^{\infty}$ written in the following form:

$$P(x) = \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n.$$

Power series are almost taken around $x_0 = 0$: if $x_0$ is not mentioned, feel free to assume that it is 0.

The definition above says that a power series is just a fancy way of writing down a sequence. This looks like it contradicts our original idea for power series, which was that we would generalize polynomials: in other words, if I give you a power series, you quite certainly want to be able to **plug numbers into it**!

The only issue with this is that sometimes, well ... you can't:

**Example.** Consider the power series

$$P(x) = \sum_{n=0}^{\infty} x^n.$$

There are values of $x$ which, when plugged into our power series $P(x)$, yield a series that fails to converge.

*Proof.* There are many such values of $x$. One example is $x = 1$, as this yields the series

$$P(x) = \sum_{n=0}^{\infty} 1,$$

which clearly fails to converge; another example is $x = -1$, which yields the series

$$P(x) = \sum_{n=0}^{\infty} (-1)^n.$$

The partial sums of this series form the sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$, which clearly fails to converge. $\square$

So: if we want to work with power series as polynomials, and not just as fancy sequences, we need to find a way to talk about where they "make sense:" in other words, we need to come up with an idea of **convergence** for power series! We do this here:

**Definition 2.2.** A power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to **converge** at some value $b \in \mathbb{R}$ if and only if the series

$$\sum_{n=0}^{\infty} a_n (b - x_0)^n$$

converges. If it does, we denote this value as $P(b)$.

The following theorem, proven in lecture, is remarkably useful in telling us where power series converge:

*Theorem* 1. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is a power series that converges at some value $b + x_0 \in \mathbb{R}$. Then $P(x)$ actually converges **uniformly** on the interval $(b - x_0, b + x_0)$ to its pointwise limit.

In particular, this tells us the following:

*Corollary* 2. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series centered at 0, and $A$ is the set of all real numbers on which $P(x)$ converges. Then there are only three cases for $A$: either

1. $A = \{0\}$,

2. $A =$ one of the four intervals $(-b, b), [-b, b), (-b, b], [-b, b]$, for some $b \in \mathbb{R}$, or

3. $A = \mathbb{R}$.

We say that a power series $P(x)$ has **radius of convergence** 0 in the first case, $b$ in the second case, and $\infty$ in the third case.

A question we could ask, given the above corollary, is the following: can we actually get all of those cases to occur? I.e. can we find power series that converge only at 0? On all of $\mathbb{R}$? On only an open interval?

To answer these questions, consider the following examples:

## 2.2 Power Series: Examples

**Example.** The power series

$$P(x) = \sum_{n=1}^{\infty} n! \cdot x^n$$

converges when $x = 0$, and diverges everywhere else.

*Proof.* That this series converges for $x = 0$ is trivial, as it's just the all-0 series.

To prove that it diverges whenever $x \neq 0$: pick any $x > 0$. Then the ratio test says that this series diverges if the limit

$$\lim_{n\to\infty} \frac{(n+1)!x^{n+1}}{n! \cdot x^n} = \lim_{n\to\infty} x(n+1) = +\infty$$

is $> 1$, which it is. So this series diverges for all $x > 0$. By applying our theorem about radii of convergence of power series, we know that our series can only converge at 0: this is because if it were to converge at any negative value $-x$, it would have to converge on all of $(-x, x)$, which is a set containing positive real numbers. □

**Example.** The power series

$$P(x) = \sum_{n=1}^{\infty} x^n$$

converges when $x \in (-1, 1)$, and diverges everywhere else.

*Proof.* Take any $x > 0$, as before, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}}{x^n} = x.$$

So the series diverges for $x > 1$ and converges for $0 \leq x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$: in our earlier discussion of power series, we proved that $P(x)$ diverged at both 1 and $-1$. So this power series converges on $(-1, 1)$ and diverges everywhere else. □

**Example.** The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges when $x \in [-1, 1)$, and diverges everywhere else.

*Proof.* Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)}{x^n/n} = \lim_{n\to\infty} x \cdot \frac{n}{n+1} = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right) = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \le x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the harmonic series (which diverges) and plugging in $-1$ yields the alternating harmonic series (which converges.) So this power series converges on $[-1, 1)$ and diverges everywhere else. $\square$

**Example.** The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

converges when $x \in [-1, 1]$, and diverges everywhere else.

*Proof.* Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)^2}{x^n/n^2} = \lim_{n\to\infty} x \cdot \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right)^2 = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \le x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the series $\sum \frac{1}{n^2}$, which we've shown converges. Plugging in $-1$ yields the series $\sum \frac{(-1)^n}{n^2}$: because the series of termwise-absolute-values converges, we know that this series converges absolutely, and therefore converges.

So this power series converges on $[-1, 1]$ and diverges everywhere else. $\square$

**Example.** The power series

$$P(x) = \sum_{n=0}^{\infty} 0 \cdot x^n$$

converges on all of $\mathbb{R}$.

*Proof.* $P(x) = 0$, for any x, which is an *exceptionally* convergent series. $\square$

**Example.** The power series

$$P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges on all of $\mathbb{R}$.

*Proof.* Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

So this series converges for any $x > 0$: applying our theorem about radii of convergence tells us that this series must converge on all of $\mathbb{R}$! $\square$

This last series is particularly interesting, as you'll see later in Math 1. One particularly nice property it has is that $P(1) = e$:

**Definition 2.3.**

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Using this, we can prove something we've believed for quite a while but never yet demonstrated:

**Theorem 2.4.** *e is irrational.*

*Proof.* We begin with a (somewhat dumb-looking) lemma:

*Lemma* 3. $e < 3$.

*Proof.* To see that $e < 3$, look at $e - 2$, factor out a $\frac{1}{2}$, and notice a few basic inequalities:

$$\begin{aligned}
e - 1 - 1 = & \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right) - 1 - 1 \\
= & \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \\
= & \frac{1}{2} \cdot \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots\right) \\
< & \frac{1}{2} \cdot \left(1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots\right) \\
= & \frac{1}{2} \cdot \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right) \\
= & \frac{1}{2} \cdot (e - 1) \\
\Rightarrow \quad 4e - 2 < & e - 1 \\
\Rightarrow \quad & e < 3.
\end{aligned}$$

□

Given this, our proof is remarkably easy! Assume that $e = \frac{a}{b}$, for some pair of integers $a, b \in \mathbb{Z}, b \geq 1$. Then we have that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!} &= \frac{a}{b} \\
\Rightarrow \quad \sum_{n=0}^{\infty} \frac{b!}{n!} &= a \cdot (b-1)! \\
\Rightarrow \quad \sum_{n=0}^{b} \frac{b!}{n!} + \sum_{n=b+1}^{\infty} \frac{b!}{n!} &= a \cdot (b-1)! \\
\Rightarrow \quad \sum_{n=b+1}^{\infty} \frac{b!}{n!} &= a \cdot (b-1)! - \sum_{n=0}^{b} \frac{b!}{n!}.
\end{aligned}$$

For $n \leq b$, notice that $\frac{b!}{n!}$ is always an integer: therefore, the right-hand-side of the last equation above is always an integer, as it's just the difference of a bunch of integers. This means, in particular, that the left-hand-side $\sum_{n=b+1}^{\infty} \frac{b!}{n!}$ is *also* an integer. What integer is it?

Well: we know that

$$0 < \frac{1}{b} < \sum_{n=b+1}^{\infty} \frac{b!}{n!} = \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} \ldots,$$

so it's a positive integer.

However, we also know that because $b \geq 1$, we have

$$\begin{aligned}\sum_{n=b+1}^{\infty} \frac{b!}{n!} &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} \ldots \\ &\leq \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots \\ &= \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \\ &= e - 2 \qquad < 1.\end{aligned}$$

So, it's an integer strictly between 0 and ... 1. As there are no integers strictly between 0 and 1, this is a contradiction! – in other words, we've just proven that $e$ must be rational. $\square$

One of the main applications of power series, as we'll see next week, is that we can use Taylor's methods to turn many functions into power series! In particular, we'll show that

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots, \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots, \text{ and} \\ e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots,\end{aligned}$$

and discuss the methods that we can use to turn many different functions into power series.

One of the main reasons we want to do this, as we discussed earlier, is because integrating and differentiating polynomials is **much easier** than integrating and differentiating power series! In particular, because power series uniformly converge within their radii of convergence $R$, we know that the equations

$$\begin{aligned}\int \left( \sum_{n=0}^{\infty} a_n x^n \right) dx &= \sum_{n=0}^{\infty} \int a_n x^n dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \text{ and} \\ \frac{d}{dx} \left( \sum_{n=0}^{\infty} a_n x^n \right) &= \sum_{n=0}^{\infty} \frac{d}{dx} (a_n x^n) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}\end{aligned}$$

hold and converge for any $x \in (-R, R)$.

In other words, power series are easily differentiated and integrated as many times as we like! This is great, in that it makes our calculations easy: however, the issue is that for many functions we'd like to study, **this isn't true**. In other words, if $f$ is a function that only has one derivative, we **know** that we can't represent $f$ with a power series, because a power series is infinitely differentiable within its radius of convergence!

In our next section, we'll introduce a different kind of series of functions, that will deal with this issue:

# 3 Fourier Series

## 3.1 Definitions and Motivation

**Definition 3.1.** A **Fourier series** is a series of functions of the form

$$\frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right),$$

where $C, a_n, b_n$ are some collection of real numbers.

The first, immediate use of Fourier series is the following theorem, which tells us that they (in a sense) can approximate far more functions than power series can:

**Theorem 3.2.** *Suppose that $f(x)$ is a real-valued function such that*

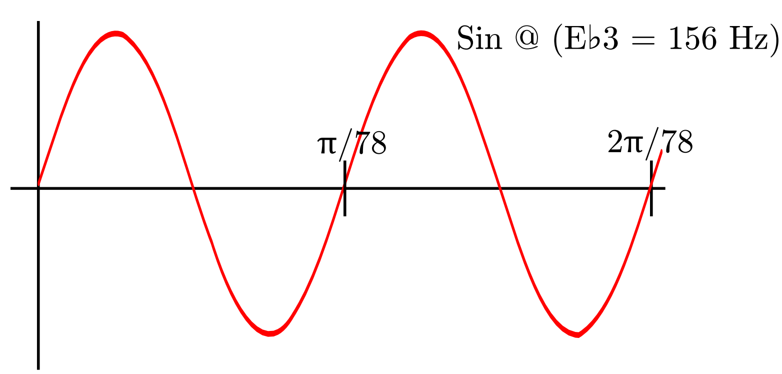
- *$f(x)$ is continuous with continuous derivative, except for at most finitely many points in $[-\pi, \pi]$.*
- *$f(x)$ is periodic with period $2\pi$: i.e. $f(x) = f(x \pm 2\pi)$, for any $x \in \mathbb{R}$.*

*Then there is a Fourier series $\frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right)$ such that*

$$f(x) = \frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right).$$

In other words, where power series can only converge to functions that are continuous and infinitely differentiable everywhere the power series is defined, Fourier series can converge to **far** more functions! This makes them, in practice, a quite useful concept, as in science we'll often want to study functions that aren't always continuous, or infinitely differentiable.

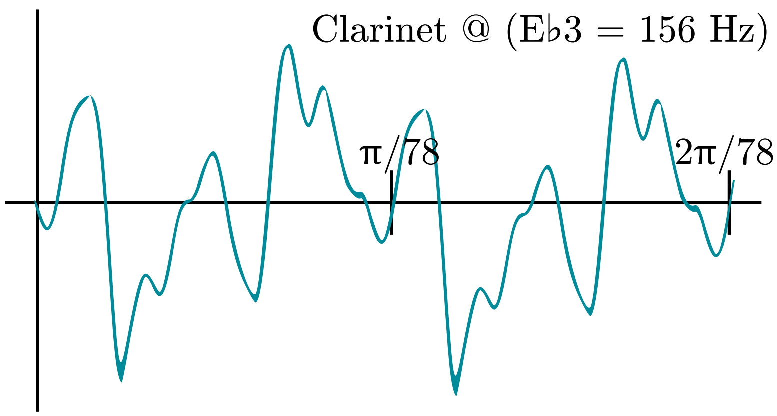
A very specific application of Fourier series is to sound and music! Specifically, recall/observe that a musical note with frequency $f$ is caused by the propogation of the longitudinal wave $\sin(2\pi f t)$ through some medium. In other words, $E\flat 3$ is just the following wave:



However, if you've ever performed in a band or listened to music, you've probably noticed that different instruments will sound quite different when playing the same note! This is because most instruments don't simply play the sine wave $\sin(2\pi f t)$, but rather play the Fourier series

$$\sum_{n=1}^{\infty} a_n \cdot \sin(2\pi f n \cdot t).$$

This is because instruments generally produce a series of **overtones**: in addition to playing the specific note chosen, they also produce sounds corresponding to all of the integer multiples of that frequency. For example, a clarinet playing $E\flat 3$ produces the following waveform:




This wave has, roughly speaking, the following Fourier series:

$$\begin{aligned}
&\sin(156 \cdot 2\pi t) + .04 \cdot \sin(312 \cdot 2\pi t) + .99 \cdot \sin(468 \cdot 2\pi t) + .12 \cdot \sin(624 \cdot 2\pi t) \\
&+ .53 \cdot \sin(780 \cdot 2\pi t) + .11 \cdot \sin(936 \cdot 2\pi t) + .26 \cdot \sin(1092 \cdot 2\pi t) + .05 \cdot \sin(1248 \cdot 2\pi t) \\
&+ .24 \cdot \sin(1404 \cdot 2\pi t) + .07 \cdot \sin(1560 \cdot 2\pi t) + .02 \cdot \sin(1716 \cdot 2\pi t) + .03 \cdot \sin(1872 \cdot 2\pi t).
\end{aligned}$$

A common task, when creating a computer synthesizer to simulate various musical instruments, is to record the waveform for a given instrument and break it down into a Fourier series, which the synthesizer can then use to "simulate" the sound of a given instrument.

### 3.2 How to Find a Fourier Series: Theory

So: the above section has hopefully motivated a little bit of the "why" behind Fourier series. Here, we'll talk about the "how:" i.e. given a periodic function $f$, how do we find its Fourier series?

The answer here is a rather strange one: **vector spaces**! In specific, look at the vector space with basis given by the **functions**

$$\left\{\frac{1}{2}\right\} \cup \{\sin(nx)\}_{n=1}^{\infty} \cup \{\cos(nx)\}_{n=1}^{\infty}.$$

Elements of this space look like linear combinations of these vectors[1]: i.e. they're of the form

$$\frac{c}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right).$$

In other words, our vector space is made out of Fourier series!

Our goal in this language, then, is to do the following: given an element $f(x)$ in our vector space, we want to find its components $c, \{a_n\}, \{b_n\}$ in every "dimension" – i.e. the components of $f(x)$ corresponding to $\frac{1}{2}$ and all of the $\sin(nx), \cos(nx)$ terms.

So: in $\mathbb{R}^3$, when we have a vector $\mathbf{v}$ that we want to break down into its component parts $(v_1, v_2, v_3)$, we do so via the **projection** operation: i.e.

$$\begin{aligned} x\text{-component of } \mathbf{v} &= \text{projection}(\mathbf{v}, (1,0,0)) \\ &= \mathbf{v} \cdot (1,0,0) \\ &= (v_1, v_2, v_3) \cdot (1,0,0) \\ &= v_1 \cdot 1 + v_2 \cdot 0 + v_3 \cdot 0 \\ &= v_1. \end{aligned}$$

Fourier's brilliant idea[2] was to define this idea of projection for our space of functions as well! In specific, consider the following definition:

$$\begin{aligned} \text{projection}\left(f(x), \sin(nx)\right) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx) dx \\ \text{projection}\left(f(x), \cos(nx)\right) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx \\ \text{projection}\left(f(x), \frac{1}{2})\right) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{2} dx. \end{aligned}$$

Why are these definitions useful? Well, if you're Fourier, you made them because you've proved the following crazy/crazy-useful **orthogonality relations**:

$$\int_{-\pi}^{\pi} \left(\frac{1}{2}\right) dx = \pi, \quad \int_{-\pi}^{\pi} \sin^2(nx) dx = \pi, \quad \int_{-\pi}^{\pi} \cos^2(nx) dx = \pi, \forall n \in \mathbb{N}.$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx) dx = 0, \quad \int_{-\pi}^{\pi} \cos(nx)\cos(mx) dx = 0, \forall n \neq m \in \mathbb{N}.$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx) dx = 0, \forall n, m \in \mathbb{N}.$$

$$\int_{-\pi}^{\pi} \sin(nx) \cdot \frac{1}{2} dx = 0, \quad \int_{-\pi}^{\pi} \cos(nx) \cdot \frac{1}{2} dx = 0, \forall n > 0 \in \mathbb{N}.$$

[1]Technically speaking, vector spaces only allow *finite* linear combinations of basis elements; so we're really working in something that's just vector-space-like. For our purposes, however, it has all of the vector space properties we're going to need, so it's a lot better for your intuition to just think of this as a vector space and not worry about the infinite-sum thing for now.

[2]Well, one of Fourier's many brilliant ideas. He had a lot.

What do these relations have to do with this strange definition of projection? Well, let's look at the $\sin(mx)$ projection onto a Fourier series $f(x) = \frac{c}{2} + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$:

$$\begin{aligned}
\operatorname{proj}(f(x), \sin(mx)) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(mx) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{c}{2} \cdot \sin(mx) + \sum_{n=1}^{\infty} (a_n \sin(nx) \sin(mx) + b_n \cos(nx) \sin(mx) dx) \\
&= \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \frac{c}{2} \cdot \sin(mx) dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \sin(nx) \sin(mx) + b_n \cos(nx) \sin(mx)) dx \right)
\end{aligned}$$

But the orthogonality relations tell us that all of these individual integrals of the $\sin(nx)\sin(mx)$, $\cos(nx)\cos(mx), \sin(mx)/2$ terms are all 0, while the $\sin^2(mx)$ term has integral $\pi$. So, in specific, we can calculate this crazy thing, and see that it's just

$$\frac{1}{\pi}(0 + a_m \cdot \pi) = a_m.$$

In other words: projection works! I.e. if we have a Fourier series $f(x) = \frac{c}{2} + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$, we have

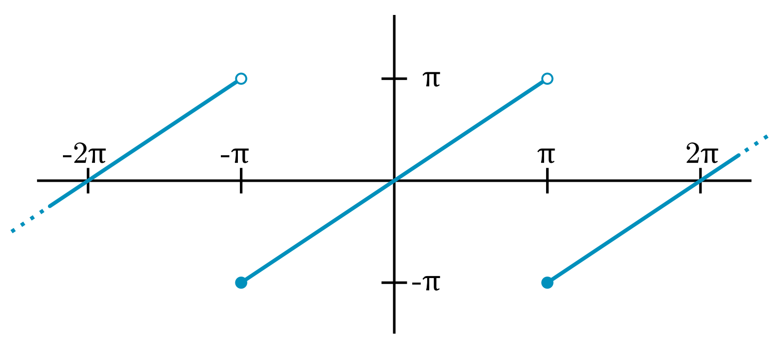
$$\begin{aligned}
&\text{projection}(f(x), \sin(nx)) = a_n, \\
&\text{projection}(f(x), \cos(nx)) = b_m, \\
&\text{projection}\left(f(x), \frac{1}{2})\right) = c.
\end{aligned}$$

So we can turn functions into Fourier series!

### 3.3 How to Find a Fourier Series: An Example

To illustrate how this works in practice, consider the following example:

**Example 3.3.** Find the Fourier series of the **sawtooth wave** $s(x)$:




$$s(x) = \begin{cases} x, & x \in [-\pi, \pi), \\ s(x \pm 2\pi), & \forall x \in \mathbb{R}. \end{cases}$$

**Solution**: We proceed via the projection method we developed above:

$$
\begin{aligned}
\text{(constant term)} &= \text{projection}\left(s(x), \frac{1}{2}\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cdot\frac{1}{2}dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\frac{1}{2}dx \\
&= \frac{1}{\pi}\left(\frac{x^2}{4}\right)\Big|_{-\pi}^{\pi} \\
&= 0. \\
\text{(}\sin(nx)\text{ term)} =& \text{projection}\left(s(x), \sin(nx)\right) \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cdot\sin(nx)dx \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)dx \\
=& \frac{1}{\pi}\left(x\cdot\frac{-\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx\right), \\
& \qquad\text{[via integration by parts with } u = x, dv = \sin(nx).\text{]} \\
=& \frac{1}{\pi}\left(\frac{\pi\cos(n\pi)+\pi\cos(-n\pi)}{n}+0\right) \\
=& \frac{2\cos(n\pi)}{n}, \quad \text{[because cos is even.]} \\
=& \frac{2(-1)^{n+1}}{n}. \\
\text{(}\cos(nx)\text{ term)} =& \text{projection}\left(s(x), \cos(nx)\right) \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cdot\cos(nx)dx \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)dx \\
=& \frac{1}{\pi}\left(x\cdot\frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nx)}{n}dx\right), \\
& \qquad\text{[via integration by parts with } u = x, dv = \cos(nx).\text{]} \\
=& \frac{1}{\pi}\left(\frac{\pi\sin(n\pi)+\pi\sin(-n\pi)}{n}+0\right) \\
=& 0.
\end{aligned}
$$

Therefore, we've proven that the Fourier series for our sawtooth wave $s(x)$ is

$$\sum_{n=1}^{\infty}\frac{2\cdot(-1)n+1\cdot\sin(nx)}{n}.$$