Math 1d	Instructor: Padraic Bartlett

Lecture 7: Generating Functions

Week 8	*Caltech - Winter 2012*

1 Random Questions

Question 1.1. *Consider the following process for generating a "random" graph on n vertices:*

- *Take n vertices.*
- *For each pair of vertices, flip a coin; if it's heads, put an edge between them, and if it's tails do not connect them with an edge.*

Show that there are $\frac{n(n-1)}{2}$ many possible results of this process.

Suppose that you consider the same process, but now on $\mathbb{N}$-many vertices. How many graphs will occur with strictly positive probability[1]*?*

2 Generating Function and Dice

2.1 Generating Functions: An Introduction

Throughout the last few weeks of our class, we've repeatedly used our knowledge of **sequences** to study **power series**. In other words, our proofs have looked like the following:

$$\left(\text{knowledge of } \{a_n\}_{n=1}^{\infty}\right) \Rightarrow \left(\text{knowledge of } \sum_{n=1}^{\infty} a_n x^n\right).$$

We've done this because, for the last few weeks, we've understood sequences *better* than we understood power series. However, this is no longer necessarily true! With our knowledge of power series and Taylor series, we have a large library of tools with which to study power series.

Given this new state of affairs, it's perhaps natural to ask if we can now **reverse** the methods we described above. In other words: suppose that we have a sequence that we want to study. What if we turned it into a power series, and used our knowledge of how that power series works to answer questions about the original series? I.e. can we make proofs that look like

$$\left(\text{knowledge of } \sum_{n=1}^{\infty} a_n x^n\right) \Rightarrow \left(\text{knowledge of } \{a_n\}_{n=1}^{\infty}\right)?$$

The answer to this question is a resounding **yes**! In mathematics, this process is called the method of **generating functions**. A brief outline for how a generating function proof goes is the following:

[1] I.e. if you pick a random integer from $\mathbb{Z}$, the odds that the integer you picked is 1 is precisely 0: i.e. the probability that you picked 1 is 0, which is not strictly positive. However, the odds that you picked a positive integer are 1/2, which *is* a strictly positive probability.

- Take some sequence $\{a_n\}_{n=1}^{\infty}$ that you want to study.
- Look at the associated power series $\sum_{n=1}^{\infty} a_n x^n$.
- Find a nice closed form (i.e. like $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$) for this power series, using your knowledge of Taylor series and power series.
- Use this closed form somehow to regain information about your original sequence. I.e. your closed form may have a different expansion that you can figure out, via Taylor series: therefore, because power series are unique, you know that the terms in this different expansion have to be equal to the terms $\sum_{n=1}^{\infty} a_n x^n$ in your original expansion! In other words, you've found new information about your sequence $\{a_n\}_{n=1}^{\infty}$!

A question two weeks ago on the HW, about Fibonacci numbers, was secretly a generating functions question in disguise! We revisit it here, and rephrase our solution in the language that we described above:

Example. Recall, from the first problem set, the Fibonacci sequence:

$$f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}, \forall n \geq 2.$$

Using the method of generating functions, can we find a closed-form expression for the elements f_n: i.e .a way of calculating f_n without having to find out what f_{n-1} and f_{n-2} are?

Answer: Let's use the method of **generating functions**! Specifically, let's look at the power series

$$\sum_{n=0}^{\infty} f_n x^n.$$

The only thing we know about the constants f_n, at first, is their recurrence relation $f_n = f_{n-1} + f_{n-2}$. So: let's plug that in to our power series! Specifically, let's plug that into all of the terms f_n with $n \geq 2$, as those are the terms where this recurrence relation holds:

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n x^n &= f_0 \cdot x^0 + f_1 \cdot x^2 + \sum_{n=2}^{\infty} f_n x^n \\
&= 0 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n \\
&= x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \\
&= x + x \sum_{n=2}^{\infty} f_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} f_{n-2} x^{n-2} \\
&= x + x \sum_{n=1}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n
\end{aligned}$$

where we justfied this last step by just shifting our indices (i.e. the sum starting at 2 of $f_{n-1}x^{n-1}$ is just the sum starting at 1 of $f_n x^n$.) Finally, if we notice that because $f_0 = 0$, we have $x\sum_{n=1}^{\infty} f_n x^n = x\sum_{n=0}^{\infty} f_n x^n$, we finally have

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n x^n &= x + x\sum_{n=0}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n \\
\Rightarrow \sum_{n=0}^{\infty} f_n x^n - x\sum_{n=0}^{\infty} f_n x^n - x^2 \sum_{n=0}^{\infty} f_n x^n &= x \\
\Rightarrow (1 - x - x^2)\sum_{n=0}^{\infty} f_n x^n &= x \\
\Rightarrow \sum_{n=0}^{\infty} f_n x^n &= \frac{x}{1 - x - x^2}.
\end{aligned}$$

Sweet! A closed form. So: according to our blueprint, we want to use this closed form to find information about our original series, possibly by finding another way to expand it.

Well: if we use partial fractions, we can see that (via algebra that you can check!)

$$\begin{aligned}
1 - x - x^2 &= (1 - xr_+) \cdot (1 - xr_-) \qquad \left(\text{where } r_+ = \frac{1+\sqrt{5}}{2}, r_- = \frac{1-\sqrt{5}}{2}\right) \\
\Rightarrow \frac{x}{1 - x - x^2} &= \frac{x}{(1 - xr_+) \cdot (1 - xr_-)} \\
&= \frac{1}{r_+ - r_-} \cdot \left(\frac{1}{1 - xr_+} - \frac{1}{1 - xr_-}\right) \\
&= \frac{1}{\sqrt{5}} \cdot \left(\frac{1}{1 - xr_+} - \frac{1}{1 - xr_-}\right) \\
&= \frac{1}{\sqrt{5}} \cdot \left(\sum_{n=0}^{\infty} (xr_+)^n - \sum_{n=0}^{\infty} (xr_-)^n\right) \\
&= \frac{1}{\sqrt{5}} \cdot \left(\sum_{n=0}^{\infty} (r_+^n - r_-^n) x^n\right)
\end{aligned}$$

So: we found a new way to expand our series! In particular, because power series are unique, we know that the coefficients of this different way to expand our series must be the same as the coefficients of our original power series $\sum f_n x^n$:

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n x^n &= \frac{1}{\sqrt{5}} \cdot \left(\sum_{n=0}^{\infty} (r_+^n - r_-^n) x^n\right) \\
\Rightarrow f_n &= \frac{r_+^n - r_-^n}{\sqrt{5}}.
\end{aligned}$$

So we have a closed form for the f_n's. In other words, it worked!

The rest of this lecture is devoted to studying a specific and particularly beautiful example of this method: the study of **nonstandard dice**!

2.2 Nonstandard Dice

Definition. Define a k-sided die as a k-sided shape on which symbols $s_1, \dots s_k \in \mathbb{N}^+$ are drawn. Analogously, we can define a k-die to be a bucket with k balls in it, each stamped with a symbol $s_i \in \mathbb{N}^+$. In this sense, "rolling" our die corresponds to picking a ball out of our bucket; for intuitive purposes, pick whichever model makes more sense and feel free to use it throughout this lecture.

For our lecture, we restrict all of our symbols to be positive integers: i.e. elements from the set $\{1, 2, 3, 4, \dots\}$.

A **standard** k-sided die D is just a k-sided die with faces $\{1, 2, 3 \dots k\}$. For example, a standard 6-die is just the normal 6-sided dice that you play most board games with.

The motivating question of this lecture is the following:

Question 1. Can you find two 6-sided dice B, C with the following property: for any n, the probability that rolling B and C together and summing them yields n is the same as the probability that rolling two standard 6-sided dice together and summing them yields n?

For example, the probability that $(B + C = 7)$ would have to be $\frac{6}{36}$, because there are 36 different ways for a pair of two 6-sided dice to be rolled, and there are precisely 6 different ways for a pair of standard 6-sided dice to sum to 7. Similarly, the probablity for $(B + C = 2)$ would have to be $\frac{1}{36}$, because there's only one way for a pair of standard 6-sided dice to sum to 2.

To answer this, surprisingly, we need to use the language of generating functions[2]! To do this, let's use the following method of turning dice into sequences:

Definition. Given a k-sided die D, let d_n denote the number of ways in which rolling D yields a n. In this sense, the die D and the sequence $\{d_n\}_{n=1}^{\infty}$ are equivalent.

For a standard k-die D, the associated sequence $\{d_n\}_{n=1}^{\infty}$ is just

$$\underbrace{1, 1, 1 \dots 1}_{\text{k 1's}}, 0, 0, \dots$$

Question 2. Take two dice $B = \{b_n\}_{n=1}^{\infty}, C = \{c_n\}_{n=1}^{\infty}$, and let

$$d_n = \text{the number of ways that rolling } B, C \text{ and summing yields } n.$$

What is $\{d_n\}_{n=1}^{\infty}$ in terms of the coefficients b_n, c_n?

Answer: How many ways can rolling B, C and summing give you n? Well: suppose you've already rolled B and gotten a k. Then you need to roll a $n - k$ on C to get a sum of n! In other words,

$$\begin{aligned} d_n &= \text{the number of ways that rolling } B, C \text{ and summing yields } n \\ &= \sum_{k=1}^{n} (\text{ways to roll } B \text{ and get } k) \cdot (\text{ways to roll } C \text{ and get } n - k) \\ &= \sum_{k=1}^{n} b_k c_{n-k}. \end{aligned}$$

[2]Well, you could try brute force and checking all 10^{12} possible pairs of dice with faces from $\{1, \dots 11\}$, but that would make for a very long and boring lecture.

So: let $A = \{a_n\}_{n=1}^{\infty} = \{1,1,1,1,1,1,0,0\ldots\}$ be a standard 6-sided die. In the language of sequences, then, we're trying to find a pair of dice-sequences $\{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ such that for every n, we have

$$\sum_{k=1}^{n} b_k c_{n-k} = \sum_{k=1}^{n} a_k a_{n-k}.$$

This looks...awful, right? In other words, we have a problem, and in the language of sequences, it's terrible. So: let's use the method of generating functions to study these sequences! After all, they can't get much worse ...

Question 3. If $A = \{a_n\}_{n=1}^{\infty}$ is a standard k-die, what is the power series $\sum_{n=1}^{\infty} a_n x^n$ associated to A?

Answer: As mentioned earlier, we have

$$\{a_n\}_{n=1}^{\infty} = \{\underbrace{1,1,1\ldots 1}_{\text{k 1's}}, 0, 0, \ldots\}.$$

Therefore, the associated power series to this sequence is just the polynomial

$$x + x^2 + x^3 + \ldots + x^k.$$

Notice that any power series associated to a k-sided dice D is just a polynomial, as any k-sided dice has only finitely many faces, and therefore finitely many nonzero elements in its associated sequence $\{d_n\}_{n=1}^{\infty}$.

Question 4. Let $B = \{b_n\}_{n=1}^{\infty}, C = \{c_n\}_{n=1}^{\infty}$, be a pair of dice, and let $B(x) = \sum b_n x^n, C(x) = \sum c_n x^n$ be their associated power series.

Let $\{d_n\}$ be the sequence associated to rolling both B, C and summing the result, as discussed before. What is the power series associated to $\{d_n\}$?

Answer: If we use our earlier observation about how we can formulate the d_n's in terms of the b_n, c_n's, we have

$$\sum_{n=1}^{\infty} d_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_k c_{n-k} \right) x^n.$$

But this is just the product of the two polynomials $B(x), C(x)$! Specifically, you can check by multiplying terms out via FOIL that

$$\left(\sum_{n=1}^{\infty} b_n x^n \right) \cdot \left(\sum_{n=1}^{\infty} c_n x^n \right) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} b_k c_{n-k} \right) x^n,$$

and therefore that

$$\begin{aligned} \sum_{n=1}^{\infty} d_n x^n &= \left(\sum_{n=1}^{\infty} b_n x^n \right) \cdot \left(\sum_{n=1}^{\infty} c_n x^n \right) \\ &= B(x) \cdot C(x). \end{aligned}$$

In other words, to get the generating function for the sum of two dice, we can simply take the product of their individual generating functions!

So, in the language of generating functions, our question is now the following:

Question 2.1. *Find a pair of polynomials with integer coefficients $B(x), C(x)$ such that*

- *$B(x), C(x)$ both correspond to 6-sided dice: i.e. $B(0) = C(0) = 0$ [no 0-faces], $B(1) = \sum b_i = 6, C(1) = \sum c_i = 6$ [they're 6-sided], and all of the coefficients of $B(x), C(x)$ are positive [you can't have a negative number of ways to roll a certain result.]*
- *Rolling B, C and summing is equivalent to rolling two standard 6-sided dice and summing: i.e. via our earlier work*

$$\begin{aligned} B(x) \cdot C(x) &= (\textit{rolling } B, C \textit{ and summing, interpreted as a polynomial}) \\ &= (\textit{rolling 2 standard 6-dice and summing, interpreted as a polynomial}) \\ &= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2. \end{aligned}$$

- *Neither B or C are standard dice: i.e. neither $B(x)$ or $C(x)$ are equal to $x + x^2 + x^3 + x^4 + x^5 + x^6$.*

Now our question is just one about algebra! I.e. we're just looking for a pair of polynomials whose product is some specific polynomial, whose coefficients are all positive, and that when you plug in 0 yield 0 and when you plug in 1 yield 6. This is **doable**!

Specifically: after playing around with the above polynomial, or talking to an algebraicist, you'll realize that

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 = (x)^2(x + 1)^2(x^2 + x + 1)^2(x^2 - x + 1)^2.$$

More specifically, none of the terms $(x), (x + 1), (x^2 + x + 1), (x^2 - x + 1)$ can be broken up into smaller polynomials, and there is no way to break up this polynomial into different integer polynomials. (In this sense, these polynomials $(x), (x+1), (x^2+x+1), (x^2-x+1)$ are thought of as **irreducible** polynomials: you cannot break them into smaller parts, and you cannot break anything made of these polynomials into different parts that does not use them. A good analogy here is to the role of **prime numbers** in the integers: just like any number can be broken up into a bunch of prime factors, any integer polynomial can be broken up into a bunch of irreducible factors.)

So: the only thing for us to do now is find out if we can split these factors $(x), (x + 1), (x^2 + x + 1), (x^2 - x + 1)$ into two polynomials, so that they both correspond to 6-sided nonstandard dice.

Because $x + 1$ is 2 at $x = 1$, $x^2 + x + 1$ is 3 at $x = 1$, and $x^2 - x + 1$ is 1 at $x = 1$, we know that each $A_i(x)$ has to have exactly one copy of both $x + 1$ and $x^2 + x + 1$ in it in order for $A_i(1)$ to be 6. As well, because they both need to be 0 at $x = 0$, we need to give each polynomial a copy of x. Consequently, the only way we can have both of these dice not be standard is if

$$\begin{aligned} B(x) &= x(x + 1)(x^2 + x + 1)(x^2 - x + 1)^2 = x^8 + x^6 + x^5 + x^4 + x^3 + x, \\ C(x) &= x(x + 1)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x; \end{aligned}$$

i.e. we have one die with faces $\{8, 6, 5, 4, 3, 1\}$ and one die with faces $\{4, 3, 3, 2, 2, 1\}$.

Check this: they actually work! For example, there are precisely 6 ways in which rolling these two dice yields 7, just like for a pair of standard 6-sided dice.

So, yeah: dice! Basically, if you take anything away from this course, it should be that series can do **everything**.