| Math 1d | Instructor: Padraic Bartlett |
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# Homework 1

*Due date: Wednesday, January 23.* — *Caltech 2013*

**Instructions**: There are **three** sections to this homework. In the first section, complete **both of the two problems** listed. In the second section, choose **two** out of the listed five to complete. Finally, in the third section, choose **one** out of the listed four to complete. If you attempt more than the listed number of problems, your highest scores from each section will be used in determining your grade. If you find yourself stuck, frustrated, or spending much more than six hours on the total problem set, contact me!

For the first section: the only resources allowed are your own notes, the online notes, and textbooks. Collaboration is not allowed on these two problems.

For the second and third sections: you are allowed to use Wikipedia, Mathematica, your notes, the online class notes, textbooks, your classmates, and other Caltech students. All of your work should, however, be written up in your own words, and you should understand your proofs well (i.e. if someone else were to ask you how to do this problem, you should be able to teach them your solution and persuade them that your methods are correct.) If you've completed a problem via Mathematica or writing a program, **you must attach your code** to receive credit.

**Section One**:

1. Find the following limits. Prove that your answer is correct.

   (a)

$$\text{(a)} \qquad \lim_{n\to\infty} \frac{n}{n+24}.$$

$$\text{(b)} \qquad \lim_{n\to\infty} \frac{\sin(n)}{n^2+\cos(n)}.$$

$$\text{(c)} \qquad \lim_{n\to\infty} a_n,$$

   where $\{a_n\}_{n=1}^{\infty}$ is the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

   formed by setting $a_1 = \sqrt{2}$ and letting $a_{n+1}$ be defined recursively as $\sqrt{2a_n}$.

2. Determine whether the following series converge or diverge. Prove your claim.

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3 - \cos(n)}.$$

$$\text{(b)} \quad \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}.$$

**Section Two**:

1. The Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ is defined as follows:

$$f_0 = 0, f_1 = 1, f_{n+2} = f_n + f_{n+1}.$$

   (a) Write the first 10 terms of the Fibonacci sequence.

   (b) Using induction, prove that

$$f_n = \frac{\varphi^n - (-1/\varphi)^n}{\sqrt{5}},$$

   where $\varphi = \frac{1+\sqrt{5}}{2}$ denotes the **golden ratio**. (Hint: notice that $\varphi^2 = \varphi + 1$, and similarly that $(-1/\varphi)^2 = (-1/\varphi) + 1$, and use this in your induction step.)

2. For any positive integer $k$, the $k$-**hailstone** sequence $\{h_n\}_{n=0}^{\infty}$ is defined as follows:

   - Define $h_0 = k$.
   - If $h_n$ is odd, define $h_{n+1} = 3h_n + 1$.
   - If $h_n$ is even, define $h_{n+1} = \frac{h_n}{2}$.

   For example, the following sequence is the 13-hailstone sequence:

$$13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots$$

   The **Collatz conjecture**, an open problem in mathematics, is the claim that no matter what number you start with, this sequence will eventually reach 1.

   (a) Write a computer program to verify that the Collatz conjecture is true for all positive natural numbers less than 1000. Attach your code!

   (b) In our example sequence where we started at 13, we got to 1 after 9 steps, i.e. at the 9th entry of our sequence. Using your program, determine which number less than 1000 takes the most steps to get to 1. How many steps does it take? (Again, attach your code or other proof.)

3. Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ converges, then it has an upper and a lower bound.

4. (a) Pick two distinct positive integers greater than 42. Write them down here.

(b) Take your two integers, and label them $x$ and $y$, respectively. Find the following limits:

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{x^n - y^n}{x^n + y^n}.$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \left(x^n + y^n\right)^{1/n}.$$

5. The following is the first five entries in the **look-and-say** sequence:

$$\begin{aligned} &1, \\ &11, \\ &21, \\ &1211, \\ &111221, \ldots \end{aligned}$$

To generate the next entry of the "look-and-say" sequence from the most recent entry, simply read the last entry aloud, counting the number of digits in groups of that digit. For example, starting from 111221, we would read

- 111 is read off as"three ones," i.e. 31.
- 22 is read off as "two twos," i.e. 22.
- 1 is read off as "one one," i.e. 11.

So the next entry in our sequence is 312211.

(a) Write the next three entries of the look-and-say sequence.

(b) Prove that no element of the look-and-say sequence will ever contain a 4.

**Section Three**:

1. In class, we showed that the series $\sum \frac{1}{n}$ diverges. Show that the sum

$$\sum_{\substack{n\in\mathbb{N}: \\ n \text{ has no } 9 \\ \text{in its digits}}} \frac{1}{n}$$

converges to some value $< 80$.

2. Suppose that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that the series $\sum_{n=1}^{\infty} a_n^2, \sum_{n=1}^{\infty} b_n^2$ both converge. Prove that the series

$$\sum_{n=1}^{\infty} (a_n \cdot b_n)$$

also converges.

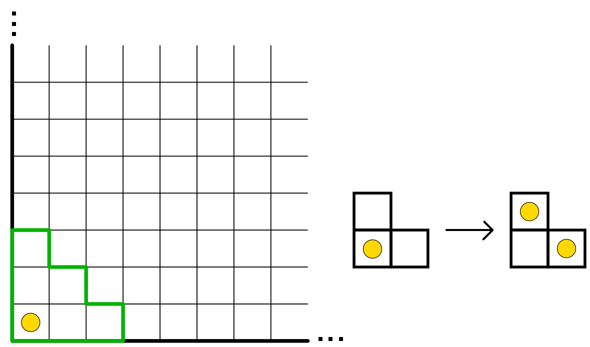
3. Does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

converge? Prove your claim.

4. Suppose you have a $\mathbb{N} \times \mathbb{N}$ grid of $1 \times 1$ squares. Consider the following game you can play on this board:

   - Starting configuration: put one coin on the square in the bottom-right-hand corner of our board.
   - Moves: suppose that there is a coin on the board such that the squares immediately to its north and east are empty. Then a valid move is the following: remove this coin from the board, and then put one new coin on the north square and another new coin on the east square.



Is it possible to get all of the coins out of the green region above in a finite number of moves? Or will there always be coins in that region, no matter what you do?

Hint: one common way that mathematicians try to study games or models like this one is by finding an **invariant**, i.e. a function or quantity that we can associate to our system that doesn't change after moves are made. For example:

(a) Try to create a way to assign "weights" to every tile, so that the total weight of tiles with coins on them is an **invariant**. In other words, find a way to associate weights $w(i, j)$ to every tile $(i, j)$, so that the sum of these weights over all of the coin-containing tiles doesn't change when we perform a move. (Try using negative powers of 2!)

(b) What is the total weight of everything outside of the green region? What is the total weight you start with? What can you deduce from this?