Math 1d	Instructor: Padraic Bartlett

Homework 3

Due date: Wednesday, February 20. *Caltech 2013*

Instructions: There are **two** sections to this homework. In the first section, complete **both of the two problems** listed. In the second section, choose **two** out of the listed five to complete. If you attempt more than the listed number of problems, your highest scores from each section will be used in determining your grade. If you find yourself stuck, frustrated, or spending much more than six hours on the total problem set, contact me!

For the first section: the only resources allowed are your own notes, the online notes, and textbooks. Collaboration is not allowed on these two problems.

For the second section: you are allowed to use Wikipedia, Mathematica, your notes, the online class notes, textbooks, your classmates, and other Caltech students. All of your work should, however, be written up in your own words, and you should understand your proofs well (i.e. if someone else were to ask you how to do this problem, you should be able to teach them your solution and persuade them that your methods are correct.) If you've completed a problem via Mathematica or writing a program, **you must attach your code** to receive credit.

For all problems listed, **show your work.** Answers are at most 2/10 of the points for any problem: I care about your methods far more than some random numbers/polynomials.

Section One:

1. Let
$$f(x) = \ln(1 + x^2).$$
   (a) Find the Taylor series $T(f(x))$ of $f(x)$. (In this problem, as we always did in class, we want to expand our Taylor series around 0.)
   (b) Look at the power series given by $T(f(x))$. What is its radius of convergence?
   (c) Where is $T(f(x))$ equal to $f(x)$?

2. Use Taylor series to approximate $\sin(3/4)$ to within $\pm .001$.

Section Two:

1. Write me a computer program that uses Taylor series to calculate $\sin(x)$ to arbitrary precision. Specifically, it should do the following:
   (a) It should prompt the user for a value of x in $[0, 2\pi]$, and a desired accuracy ϵ.
   (b) It should then call one function that (given x, ϵ) calculates a value of n such that $R_n(\sin(x)) < \epsilon$.
   (c) Using that argument, it should call a second function that (given x, n) returns $T_n(\sin(x))$.
   (d) It should then output $T_n(\sin(x))$ and n.

   As a test of your program, find $\sin(2)$ to within $\pm 10^{-5}$.

   (Note: Programs that simply invoke an existing $\sin(x)$ function in whatever language you're coding in will receive no points and/or be set on fire.)

2. Like $\int e^{-x^2}dx$, the indefinite integral

$$\int \frac{\sin(x)}{x}dx$$

cannot be expressed as some finite combination of the elementary functions {sin, cos, e^x, ln,polynomials}: i.e. there is no way to write the antiderivative of $\frac{\sin(x)}{x}$ by any combination of such elementary functions.

Despite this, use Taylor series to approximate the definite integral

$$\int_1^3 \frac{\sin(x)}{x}dx$$

to within $\pm .01$ of its exact value. (Hint: mimic the example in lecture where we approximated the integral $\int_{1/2}^{1/2} e^{-x^2}dx$.)

3. (a) Using induction and also integration by parts and an inductive argument, show that

$$\int_0^\infty e^{-at}\cdot t^n dt = \frac{n!}{a^{n+1}}.$$

holds for any $a > 0$.

(b) Using the Taylor series for $\cos(t)$ and part (a), show that for $a > 1$, we have

$$\int_0^\infty e^{-at}\cos(t)dt = \frac{a}{a^2+1}.$$

(Hint: By using (a) and $\cos(t)$, you'll be able to turn this integral into a power series evaluated at $\frac{1}{a^2}$. What power series is this?)

4. Sometimes, the Taylor series of a function can be a good approximation to a function even where it **fails to converge**! Consider the following example:

$$G(x) = \int_0^\infty e^{-x\cdot t}\cdot\frac{1}{1+t}dt.$$

If we were to attack this integral like problem 3, we would want to expand $\frac{1}{1+t}$ into its power series

$$\frac{1}{1+t} = \sum_{n=0}^\infty (-1)^n\cdot t^n = 1 - t + t^2 - t^3 \ldots$$

However, we know that this series is only convergent for $|t| < 1$, and we're integrating on $(0, \infty)$.

(a) Despite this, do it anyways! I.e. show that the integral

$$\int_0^\infty e^{-x\cdot t}\cdot\left(1 - t + t^2 - t^3 \ldots\right)dt.$$

is the series

$$P(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5}\cdots$$

(Hint: you are allowed to use 3(a) without proof. Do so.)

(b) Show that this series diverges for any nonzero $x \in \mathbb{R}$.

(c) However, surprisingly enough, we can still use this power series to roughly calculate $G(x)$, for lots of values of x! I.e. it is known that $G(10)$ is about .0915633. Using Mathematica/matlab/a calculator, find the partial sums $P_n(10)$ of the first n terms in this power series evaluated at 10, for n between 1 and 14. Which value(s) of n are the closest to the actual value of this integral?

5. (a) Suppose that $P(x) = \sum_{n=0}^{\infty} a_n x^n$ is an odd[1] function. Show that a_n is 0 whenever n is even.

(b) Suppose that $P(x) = \sum_{n=0}^{\infty} a_n x^n$ is an even[2] function. Show that a_n is 0 whenever n is odd.

[1] I.e. $P(x) = -P(-x)$.
[2] I.e. $P(x) = P(-x)$.