Math 1d	Instructor: Padraic Bartlett

Homework 4

Due date: Wednesday, Feb. 27. *Caltech 2013*

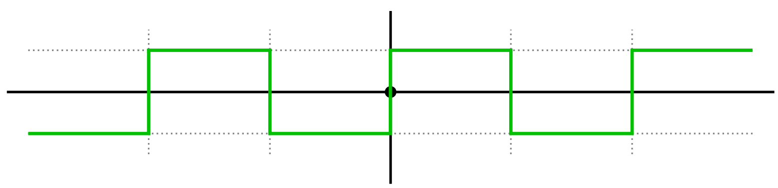
Instructions: This is a little different than normal. There is **one** section to this homework. You should solve **any two problems** in this section. If you attempt more than the listed number of problems, your highest scores from each section will be used in determining your grade. If you find yourself stuck, frustrated, or spending much more than six hours on the total problem set, contact me!

You are allowed to use Wikipedia, Mathematica, your notes, the online class notes, textbooks, your classmates, and other Caltech students. All of your work should, however, be written up in your own words, and you should understand your proofs well (i.e. if someone else were to ask you how to do this problem, you should be able to teach them your solution and persuade them that your methods are correct.) If you've completed a problem via Mathematica or writing a program, **you must attach your code** to receive credit.

For all problems listed, **show your work.** Any problems that involve code should attach code and also email me a copy so I can run it! Answers are at most 2/10 of the points for any problem: I care about your methods far more than some random numbers/polynomials.

Section One:

1. (a) Find the Fourier series of the **square wave** $s(x)$:



$$s(x) = \begin{cases} -1, & x \in [-\pi, 0), \\ 1, & x \in [0, \pi), \\ s(x \pm 2\pi), & \forall x \in \mathbb{R}. \end{cases}$$

   (b) Using (Mathematica/Maple/Matlab/Wolfram Alpha/your favorite program), graph the sum of the first hundred terms of the Fourier series from $-\pi$ to π, and attach your graph. Does it look like a square wave? What does it look like is occuring near the "jump discontinuities" at $-\pi, 0$, and π? (This doesn't have to be rigorous: just describe visually what you see. If you **do** want a rigorous discussion of what's going on here, look up the Gibbs phenomenon on Wikipedia.)

2. Using Mathematica's Play function and the harmonic analysis at http://hyperphysics.phy-astr.gsu.edu/hbase/music/flutew.html, create a Fourier series that sounds like a flute playing $F4$ when played in Mathematica. (If you are unsure what this means, ask me!)

3. Let z_1, z_2, z_3 be three elements of $\mathbb{C}$ such that

   - $z_1 + z_2 + z_3 = 0$, and
   - $|z_1| = |z_2| = |z_3|$.

Show that these three points, as graphed in $\mathbb{C}$, form the vertices of an equilateral triangle. (Hint: use polar coordinates: i.e. write $z_k = r_k \cdot e^{i\theta_k}$. What do you know about the r_i's?)

4. A **n^{th}-root of unity** is a complex number $\omega \in \mathbb{C}$ such that $\omega^n = 1$. For example, -1 and 1 are the only two 2^{nd}-roots of unity, because they are the only two complex numbers ω such that $\omega^2 = 1$.

   (a) Find all of the third roots of unity. (Hint: use polar coördinates. In other words, think of elements of $\mathbb{C}$ as numbers of the form $re^{i\theta}$.)

   (b) Let ω be a n^{th}-root of unity not equal to 1. Show that

   $$\sum_{k=0}^{n-1} \omega^k = 0.$$

   (Hint: Draw all of these powers of ω on the complex plane for some small values of n, like 3 or 4. Graphically, why is this sum 0?)

5. The complex numbers $\mathbb{C}$ were formed by taking the real numbers $\mathbb{R}$ and adding the symbol i, where we thought of i as $\sqrt{-1}$.

   Similarly, we can define the **Gaussian integers** as the set

   $$\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\},$$

   formed by taking the integers and adding the symbol i.

   (a) In the integers, we know that a number is even if and only if it's a multiple of 2. Show that in the Gaussian integers, the square of the norm[1] $|\alpha|^2$ of a Gaussian integer α is even if and only if it is a multiple of $1 + i$: i.e.

   $$2 \Big| |\alpha|^2 \Leftrightarrow \alpha = (1 + i)(c + di),$$

   for some $c, d \in \mathbb{Z}$. (Hint: Feel free to use the following fact without proof: for any two complex numbers α, β, we have $|\alpha \cdot \beta| = |\alpha| \cdot |\beta|$.)

   (b) Find the Gaussian integer $\beta = c + di$ such that $(1 + i)(c + di) = 2$.

   (c) Show that if α, β are a pair of Gaussian integers such that $\alpha \cdot \beta = 1 + i$, one of α, β is equal to one of $\pm 1, \pm i$.

   In this sense, $1 + i$ is a "prime" in the Gaussian integers, in much the same sense that 2 is a prime in the normal integers.

6. Oh noes! Your prefrosh got lost somewhere on campus. Will it ever come back?

   Specifically, consider the following mathematical model for this situation:

   - Suppose that campus is modeled by $\mathbb{Z}^2$, and that the dorms are located at the point $(0, 0)$.
   - Furthermore, suppose that your prefrosh at time $t = 0$ starts at some point (a, b) in $\mathbb{Z}^2$.

[1] Recall from class: the norm $|z|$ of a complex number $z = x + iy$, $x, y \in \mathbb{R}$ is just the distance from z to the origin: i.e. $\sqrt{x^2 + y^2}$. Therefore, the square of the norm is just $x^2 + y^2$. Notice that this is always a positive integer whenever $x + iy$ is a Gaussian integer.

- Once every minute, the prefrosh selects one of the four directions {north, south, east, west} at random and travels one step in that direction.
- The prefrosh returns home and is no longer lost if it ever makes it to $(0, 0)$. Otherwise, it wanders forever.

Prove that, given enough time, your prefrosh will always return.

7. Create a simulation for problem (6)! In other words, create a computer program that does the following:

   - As inputs, it takes in a size value n, a time value t, and a starting location (a, b).
   - Using these inputs, it creates a $n \times n$ grid, and places a prefrosh at location (a, b).
   - Then, the program "randomly walks" the prefrosh around our grid t times.
   - This program should be visually interpreted as the following: display our $n \times n$ grid as a $n \times n$ grid of colored cells/pixels/etc. If a cell has never been visited by the prefrosh, it should be white. If it is entered by the prefrosh, it should turn gray. If it is visited by the prefrosh again, it should turn black. In theory, then, your program should start out by displaying a completely white grid, on which squares should turn gray and later black as the prefrosh wanders.

   (Props to Edward for the problem idea.)