Math 1d — Instructor: Padraic Bartlett

Lecture 1: Sequences

Week 1 — *Caltech 2013*

1 Sequences and Series

This week's lectures in Math 8 are going to focus on **sequences** and **convergence**. We list some basic definitions here:

1.1 Sequences: Definitions

Definition 1.1. A **sequence** of real numbers is a collection of real numbers $\{a_n\}_{n=1}^{\infty}$ indexed by the natural numbers.

Definition 1.2. A sequence $\{a_n\}_{n=1}^{\infty}$ is called **bounded** if there is some value $B \in \mathbb{R}$ such that $|a_n| < B$, for every $n \in \mathbb{N}$. Similarly, we say that a sequence is **bounded above** if there is some value U such that $a_n \leq U, \forall n$, and say that a sequence is **bounded below** if there is some value L such that $a_n \geq L, \forall n$.

Definition 1.3. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **monotonically increasing** if $a_n \leq a_{n+1}$, for every $n \in \mathbb{N}$; conversely, a sequence is called **monotonically decreasing** if $a_n \geq a_{n+1}$, for every $n \in \mathbb{N}$.

Definition 1.4. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value λ if the a_n's "go to λ" at infinity. To put it more formally, $\lim_{n\to\infty} a_n = \lambda$ iff for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n must be within ϵ of our limit λ.

In symbols:

$$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

Convergence is one of the most useful properties of sequences! If you know that a sequence converges to some value λ, you know, in a sense, where the sequence is "going," and furthermore know where almost all of its values are going to be (specifically, close to λ.)

Because convergence is so useful, we've developed a number of tools for determining where a sequence is converging to:

1.2 Sequences: Convergence Tools

1. **The definition of convergence**: The simplest way to show that a sequence converges is sometimes just to use the definition of convergence. In other words, you want to show that for any distance ϵ, you can eventually force the a_n's to be within ϵ of our limit, for n sufficiently large.

   How can we do this? One method I'm fond of is the following approach:

- First, examine the quantity $|a_n - L|$, and try to come up with a very simple upper bound that depends on n and goes to zero. Example bounds we'd love to run into: $1/n, 1/n^2, 1/\log(\log(n))$.
- Using this simple upper bound, given $\epsilon > 0$, determine a value of N such that whenever $n > N$, our simple bound is less than ϵ. This is usually pretty easy: because these simple bounds go to 0 as n gets large, there's always some value of N such that for any $n > N$, these simple bounds are as small as we want.
- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $n > N$, $|a_n - L| < \epsilon$.

2. **Arithmetic and sequences**: These tools let you combine previously-studied results to get new ones. Specifically, we have the following results:

   - *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
   - *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
   - *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

3. **Composition of sequences and functions**: Suppose that $f(x)$ is a continuous function and that $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence. Then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$. In other words, we can push continuous functions in and out of limits, as long as those limits exist.

4. **Monotone and bounded sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges. If a sequence is monotone, this is usually the easiest way to prove that your sequence converges, as both monotone and bounded are "easy" properties to work with. One interesting facet of this property is that it can tell you that a sequence converges without necessarily telling you what it converges to! So, it's often of particular use in situations where you just want to show something converges, but don't actually know where it converges *to*.

5. **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l. This is particularly useful for sequences with things like sin(horrible things) in them, as it allows you to "ignore" bounded bits that aren't changing where the sequence goes.

6. **Cauchy sequences**: We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m > n \geq N$, we have

$$|a_m - a_n| < \epsilon.$$

   You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

The Cauchy theorem, in this situation, is the following: a sequence is Cauchy if and only if it converges.

The Cauchy criterion doesn't come up as often as the others in Math 1a (later in mathematics, however, it shows up pretty much everywhere!) Its main uses are for working with series (we'll have an example of this later, and define series later as well!), and for sequences whose limits we don't know: like the monotone-bounded-convergence theorem, this result doesn't need you to know where a sequence is converging to in order to show that it converges.

1.3 Sequences: Applications of Convergence Tools

In this section, we work an example for each of these tools. We start by illustrating how to prove a sequence converges using just the definition:

Claim 1. (Definition of convergence example:)

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0.$$

Proof. When we discussed the definition as a convergence tool, we talked about a "blueprint" for how to go about proving convergence from the definition: (1) start with $|a_n - L|$, (2) try to find a simple upper bound on this quantity depending on n, and (3) use this simple bound to find for any ϵ a value of N such that whenever $n > N$, we have

$$|a_n - L| < (\text{simple upper bound}) < \epsilon.$$

Let's try this! Specifically, examine the quantity $|\sqrt{n+1} - \sqrt{n} - 0|$:

$$\begin{aligned}
|\sqrt{n+1} - \sqrt{n} - 0| &= \sqrt{n+1} - \sqrt{n} \\
&= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{\sqrt{n}}.
\end{aligned}$$

All we did here was hit our $|a_n - L|$ quantity with a ton of random algebra, and kept trying things until we got something simple. The specifics aren't as important as the idea here: just start with the $|a_n - L|$ bit, and try everything until it's bounded by something simple and small!

In our specific case, we've acquired the upper bound $\frac{1}{\sqrt{n}}$, which looks rather simple: so let's see if we can use it to find a value of N.

Take any $\epsilon < 0$. If we want to make our simple bound $\frac{1}{\sqrt{n}} < \epsilon$, this is equivalent to making $\frac{1}{\epsilon} < \sqrt{n}$, i.e $\frac{1}{\epsilon^2} < n$. So, if we pick $N > \frac{1}{\epsilon^2}$, we know that whenever $n > N$, we have $n > \frac{1}{\epsilon^2}$, and therefore that our simple bound is $< \epsilon$. But this is exactly what we wanted!

In specific, for any $\epsilon > 0$, we've found a N such that for any $n > N$, we have

$$|\sqrt{n+1} - \sqrt{n} - 0| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon,$$

which is the definition of convergence. So we've proven that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$. □

Claim 2. (Arithmetic and Sequences example:) The sequence

$$a_1 = 1,$$
$$a_{n+1} = \sqrt{1 + a_n^2}$$

does not converge.

Proof. We proceed by contradiction: in other words, suppose that this sequence does converge to some value L, say. Then, examine the limit

$$\lim_{n\to\infty} a_n^2.$$

Because squaring things is a continuous operation, we know that

$$\lim_{n\to\infty} a_n^2 = (\lim_{n\to\infty} a_n)^2 = L^2.$$

However, we can also use the recursive definition of the a_n's to see that

$$\begin{aligned}\lim_{n\to\infty} a_n^2 &= \lim_{n\to\infty} \left(\sqrt{1 + a_{n-1}^2}\right)^2 \\ &= \lim_{n\to\infty} (1 + a_{n-1}^2)\end{aligned}$$

However, we know that $\lim_{n\to\infty} a_{n-1}^2 = \lim_{n\to\infty} a_n^2 = L^2$, because the two sequences are the same (just shifted over one place) and thus have the same behavior at infinity. Therefore, we know that both $\lim_{n\to\infty} 1$ and $\lim_{n\to\infty} a_{n-1}^2$ both exist: as a result, we can apply our result on arithmetic and sequences to see that

$$\lim_{n\to\infty} (1 + a_{n-1}^2) = \left(\lim_{n\to\infty} 1\right) + \left(\lim_{n\to\infty} a_{n-1}^2\right) = 1 + L^2.$$

So, we've just shown that $L^2 = 1 + L^2$: i.e. $0 = 1$. This is clearly nonsense: so we've arrived at a contradiction. Therefore, our original assumption (that our sequence $\{a_n\}_{n=1}^{\infty}$ converged must be false – i.e. this sequence must diverge, as claimed. □

Claim 3. (Another arithmetic and sequences example:) For any two positive real numbers $x > y > 0$, show that

$$\lim_{n\to\infty} \frac{x^n - y^n}{x^n + y^n} = 1.$$

Proof. Using the fact that $0 < y < x$, write $y = cx$, for some positive real number $c < 1$. Then, our limit is just

$$\lim_{n\to\infty} \frac{x^n - (cx)^n}{x^n + (cx)^n} = \lim_{n\to\infty} \frac{x^n - c^n x^n}{x^n + c^n x^n} = \lim_{n\to\infty} \frac{x^n(1 - c^n)}{x^n(1 + c^n)} = \lim_{n\to\infty} \frac{1 - c^n}{1 + c^n}.$$

Now, notice that because $0 < c < 1$, $\lim_{n\to\infty} 1 - c^n = \lim_{n\to\infty} 1 + c^n = 1$. Because of this, we can move our limit above into the fraction (because both the top and bottom limits exist,) and get

$$\lim_{n\to\infty} \frac{1 - c^n}{1 + c^n} = \frac{\lim_{n\to\infty} 1 - c^n}{\lim_{n\to\infty} 1 + c^n} = \frac{1}{1} = 1.$$

So our original limit is 1, as claimed. □

Claim 4. (Monotone convergence theorem example:)

$$\lim_{n\to\infty} 2^{1/n} = 1.$$

Proof. Let's start by using the monotone-bounded convergence theorem to show that the sequence $\{2^{1/n}\}_{n=1}^{\infty}$ converges (without worrying about what it actually converges *to* yet.) To do this, we need to just do two things: show that our sequence is **monotone-decreasing** and that it is **bounded below.**

Monotone-decreasing: we claim that

$$2^{\frac{1}{n+1}} < 2^{\frac{1}{n}}.$$

To see this, raise the left and right-hand-sides to the power $n(n+1)$, and simplify:

$$\begin{aligned} 2^{\frac{1}{n+1}} &< 2^{\frac{1}{n}} \\ \Leftrightarrow 2^{\frac{n(n+1)}{n+1}} &< 2^{\frac{n(n+1)}{n}} \\ \Leftrightarrow 2^n &< 2^{n+1} \\ \Leftrightarrow 1 &< 2. \end{aligned}$$

So our claim is equivalent to the inequality $1 < 2$, which is trivially true: so our sequence is monotonically decreasing, as claimed.

We claim that 1 is a lower bound: i.e. that $2^{1/n} > 1$, for every n. To see this, just raise both sides to the n-th power; we can do this without disturbing our inequality because both sides are positive. This tells us that $2^{1/n} > 1$ is equivalent to the claim $2 > 1$, which we know to be true.

So our sequence is monotone and bounded: by the monotone-bounded convergence theorem, it must converge to some value L. □

Claim 5. (Continuity and sequences example:)

$$\lim_{n\to\infty} 2^{1/n} = 1.$$

Proof. First, recall from Math 1 that the function $x \mapsto 2^x$ is continuous and well-defined on the real numbers. Therefore, because the limit $\lim_{n\to\infty} \frac{1}{n}$ exists, we can use our result on continuity and sequences to say that

$$\lim_{n\to\infty} 2^{1/n} = 2^{\lim_{n\to\infty} \frac{1}{n}} = 2^0 = 1.$$

□

Claim 6. (Squeeze theorem example:)

$$\lim_{n\to\infty} \frac{\sin\left(n^2 \cdot \pi^{n^e - 12n} \cdot n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right)}{n} = 0.$$

Proof. The idea of squeeze theorem examples is that they allow you to get rid of awful-looking things whenever they aren't materially changing where the sequence is actually going. Specifically, in our example here, the sin(terrible things) part is awful to work with, but really isn't doing anything to our sequence: the relevant part is the denominator, which is going to infinity (and therefore forcing our sequence to go to 0.

Rigorously: we have that

$$-1 \leq \sin(\text{terrible things}) \leq 1,$$

no matter what terrible things we've put into the sin function. Dividing the left and right by n, we have that

$$-\frac{1}{n} \leq \frac{\sin(\text{terrible things})}{n} \leq \frac{1}{n},$$

for every n. Then, because $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, the squeeze theorem tells us that

$$\lim_{n\to\infty} \frac{\sin\left(n^2 \cdot \pi^{n^e - 12n} \cdot n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right)}{n} = 0$$

as well. □