Math 1d — Instructor: Padraic Bartlett

Lecture 4: Power Series

Week 4 — *Caltech 2013*

1 Power Series

The motivation for **power series**, roughly speaking, is the observation that polynomials are really *quite nice*. Specifically, if I give you a polynomial, you can

- differentiate and take integrals easily,
- add and multiply polynomials together and easily express the result as another polynomial,
- find its roots,

and do most anything else that you'd ever want to do to a function! One of the only downsides to polynomials, in fact, is that there are functions that **aren't** polynomials! In specific, the very useful functions

$$\sin(x), \cos(x), \ln(x), e^x, \frac{1}{x}$$

are all not polynomials, and yet are remarkably useful/frequently occuring objects.

So: it would be nice if we could have some way of "generalizing" the idea of polynomials, so that we could describe functions like the above in some sort of polynomial-ish way – possibly, say, as polynomials of "infinite degree?" How can we do that?

The answer, as you may have guessed, is via **power series**:

Definition 1.1. A **power series** $P(x)$ centered at x_0 is just a "series of functions" of the following form:

$$P(x) = \sum_{n=0}^{\infty} a_n \cdot x^n.$$

A power series is uniquely determined by its coefficients, the sequence of numbers $\{a_n\}_{n=1}^{\infty}$.

Just like with normal series, the main thing we are interested in with power series is **convergence.** Specifically, consider the easier-to-work-with case of pointwise convergence; in this situation, we are now asking for which values of x does the series of numbers $\sum a_n x^n$ converge.

Sometimes, a power series does not converge on all of its values:

Example 1.2. Consider the power series

$$P(x) = \sum_{n=0}^{\infty} x^n.$$

There are values of x which, when plugged into our power series $P(x)$, yield a series that fails to converge.

Proof. There are many such values of x. One example is $x = 1$, as this yields the series

$$P(x) = \sum_{n=0}^{\infty} 1,$$

which clearly fails to converge; another example is $x = -1$, which yields the series

$$P(x) = \sum_{n=0}^{\infty} (-1)^n.$$

The partial sums of this series form the sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$, which clearly fails to converge[1]. □

Now, suppose that we want to find **all** of the values on which a given power series converges. The above piecemeal procedure of just trying various points seems like a bad strategy; there are a lot more numbers in $\mathbb{R}$ than we have paper. Thankfully, the folllowing theorem, which you can prove using the comparison test and a little bit of work, saves us a lot of casework:

Theorem 1. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series that converges at some value $R \in \mathbb{R}$. Then $P(x)$ actually converges on every value in the interval $(-R, R)$.

In particular, if we use the comparison test, the result above gives us the following powerful corollary:

Corollary 2. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series centered at 0, and A is the set of all real numbers on which $P(x)$ converges. Then there are only six cases for A: either

1. $A = \{0\}$,
2. $A =$ one of the four intervals $(-R, R), [-R, R), (-R, R], [-R, R]$, for some $R \in \mathbb{R}$, or
3. $A = \mathbb{R}$.

We say that a power series $P(x)$ has **radius of convergence** 0 in the first case, R in the second case, and ∞ in the third case.

A question we could ask, given the above result, is the following: can we actually get all of those cases to occur? I.e. can we find power series that converge only at 0? On all of $\mathbb{R}$? On only an open interval?

To answer these questions, consider the following examples:

[1] Though it **wants** to converge to 1/2. Go to wikipedia and read up on Grandi's series for more information!

Example 1.3. The power series

$$P(x) = \sum_{n=1}^{\infty} n! \cdot x^n$$

converges when $x = 0$, and diverges everywhere else.

Proof. That this series converges for $x = 0$ is trivial, as it's just the all-0 series.

To prove that it diverges whenever $x \neq 0$: pick any $x > 0$. Then the ratio test says that this series diverges if the limit

$$\lim_{n\to\infty} \frac{(n+1)!x^{n+1}}{n! \cdot x^n} = \lim_{n\to\infty} x(n+1) = +\infty$$

is > 1, which it is. So this series diverges for all $x > 0$. By applying our theorem about radii of convergence of power series, we know that our series can only converge at 0: this is because if it were to converge at any negative value $-x$, it would have to converge on all of $(-x, x)$, which is a set containing positive real numbers. □

Example 1.4. The power series

$$P(x) = \sum_{n=1}^{\infty} x^n$$

converges when $x \in (-1, 1)$, and diverges everywhere else.

Proof. Take any $x > 0$, as before, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}}{x^n} = x.$$

So the series diverges for $x > 1$ and converges for $0 \leq x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$: in our earlier discussion of power series, we proved that $P(x)$ diverged at both 1 and -1. So this power series converges on $(-1, 1)$ and diverges everywhere else. □

Example 1.5. The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges when $x \in [-1, 1)$, and diverges everywhere else.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)}{x^n/n} = \lim_{n\to\infty} x \cdot \frac{n}{n+1} = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right) = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \leq x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the harmonic series (which diverges) and plugging in -1 yields the alternating harmonic series (which converges.) So this power series converges on $[-1, 1)$ and diverges everywhere else. □

Example 1.6. The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

converges when $x \in [-1, 1]$, and diverges everywhere else.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)^2}{x^n/n^2} = \lim_{n\to\infty} x \cdot \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right)^2 = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \le x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the series $\sum \frac{1}{n^2}$, which we've shown converges. Plugging in -1 yields the series $\sum \frac{(-1)^n}{n^2}$: because the series of termwise-absolute-values converges, we know that this series converges absolutely, and therefore converges.

So this power series converges on $[-1, 1]$ and diverges everywhere else. □

Example 1.7. The power series

$$P(x) = \sum_{n=0}^{\infty} 0 \cdot x^n$$

converges on all of $\mathbb{R}$.

Proof. $P(x) = 0$, for any x, which is an *exceptionally* convergent series. □

Example 1.8. The power series

$$P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges on all of $\mathbb{R}$.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

So this series converges for any $x > 0$: applying our theorem about radii of convergence tells us that this series must converge on all of $\mathbb{R}$! □