Math 1d — Instructor: Padraic Bartlett

Lecture 7: Fourier Series and Complex Power Series

Week 7 — *Caltech 2013*

1 Fourier Series

1.1 Definitions and Motivation

Definition 1.1. A **Fourier series** is a series of functions of the form

$$\frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right),$$

where C, a_n, b_n are some collection of real numbers.

The first, immediate use of Fourier series is the following theorem, which tells us that they (in a sense) can approximate far more functions than power series can:

Theorem 1.2. *Suppose that $f(x)$ is a real-valued function such that*

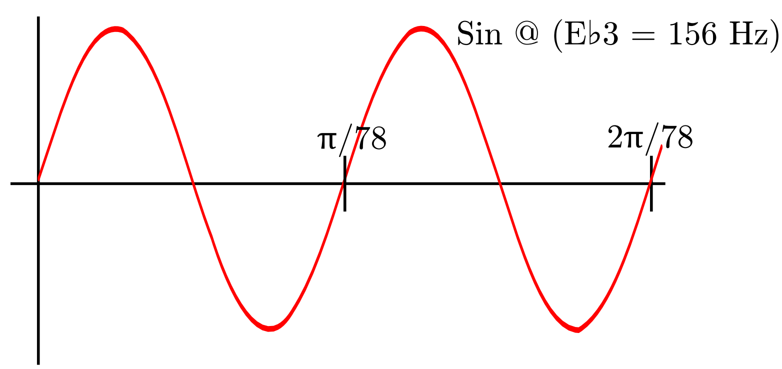
- *$f(x)$ is continuous with continuous derivative, except for at most finitely many points in $[-\pi, \pi]$.*
- *$f(x)$ is periodic with period 2π: i.e. $f(x) = f(x \pm 2\pi)$, for any $x \in \mathbb{R}$.*

Then there is a Fourier series $\frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right)$ such that

$$f(x) = \frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right).$$

In other words, where power series can only converge to functions that are continuous and infinitely differentiable everywhere the power series is defined, Fourier series can converge to **far** more functions! This makes them, in practice, a quite useful concept, as in science we'll often want to study functions that aren't always continuous, or infinitely differentiable.

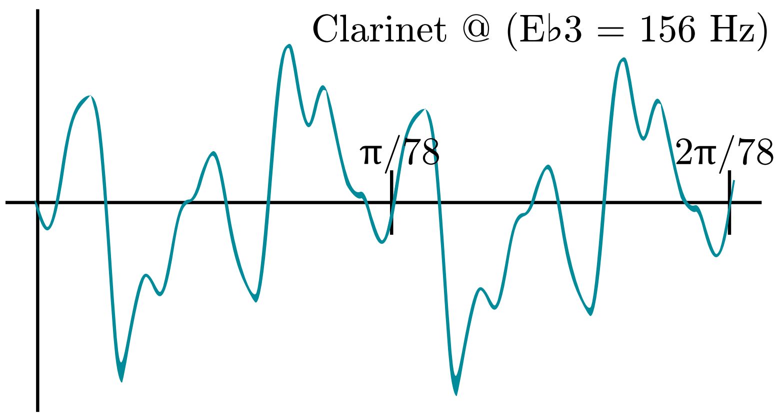
A very specific application of Fourier series is to sound and music! Specifically, recall/observe that a musical note with frequency f is caused by the propogation of the longitudinal wave $\sin(2\pi f t)$ through some medium. In other words, $E\flat 3$ is just the following wave:



However, if you've ever performed in a band or listened to music, you've probably noticed that different instruments will sound quite different when playing the same note! This is because most instruments don't simply play the sine wave $\sin(2\pi f t)$, but rather play the Fourier series

$$\sum_{n=1}^{\infty} a_n \cdot \sin(2\pi f n \cdot t).$$

This is because instruments generally produce a series of **overtones**: in addition to playing the specific note chosen, they also produce sounds corresponding to all of the integer multiples of that frequency. For example, a clarinet playing $E\flat 3$ produces the following waveform:



This wave has, roughly speaking, the following Fourier series:

$$\begin{aligned} &\sin(156 \cdot 2\pi t) + .04 \cdot \sin(312 \cdot 2\pi t) + .99 \cdot \sin(468 \cdot 2\pi t) + .12 \cdot \sin(624 \cdot 2\pi t) \\ &+ .53 \cdot \sin(780 \cdot 2\pi t) + .11 \cdot \sin(936 \cdot 2\pi t) + .26 \cdot \sin(1092 \cdot 2\pi t) + .05 \cdot \sin(1248 \cdot 2\pi t) \\ &+ .24 \cdot \sin(1404 \cdot 2\pi t) + .07 \cdot \sin(1560 \cdot 2\pi t) + .02 \cdot \sin(1716 \cdot 2\pi t) + .03 \cdot \sin(1872 \cdot 2\pi t). \end{aligned}$$

A common task, when creating a computer synthesizer to simulate various musical instruments, is to record the waveform for a given instrument and break it down into a Fourier series, which the synthesizer can then use to "simulate" the sound of a given instrument.

1.2 How to Find a Fourier Series: Theory

So: the above section has hopefully motivated a little bit of the "why" behind Fourier series. Here, we'll talk about the "how:" i.e. given a periodic function f, how do we find its Fourier series?

The answer here is a rather strange one: **vector spaces**! In specific, look at the vector space with basis given by the **functions**

$$\left\{\frac{1}{2}\right\} \cup \{\sin(nx)\}_{n=1}^{\infty} \cup \{\cos(nx)\}_{n=1}^{\infty}.$$

Elements of this space look like linear combinations of these vectors[1]: i.e. they're of the form

$$\frac{c}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right).$$

In other words, our vector space is made out of Fourier series!

Our goal in this language, then, is to do the following: given an element $f(x)$ in our vector space, we want to find its components $c, \{a_n\}, \{b_n\}$ in every "dimension" – i.e. the components of $f(x)$ corresponding to $\frac{1}{2}$ and all of the $\sin(nx), \cos(nx)$ terms.

So: in $\mathbb{R}^3$, when we have a vector $\mathbf{v}$ that we want to break down into its component parts (v_1, v_2, v_3), we do so via the **projection** operation: i.e.

$$\begin{aligned}
x\text{-component of } \mathbf{v} &= \text{projection}(\mathbf{v}, (1,0,0)) \\
&= \mathbf{v} \cdot (1,0,0) \\
&= (v_1, v_2, v_3) \cdot (1,0,0) \\
&= v_1 \cdot 1 + v_2 \cdot 0 + v_3 \cdot 0 \\
&= v_1.
\end{aligned}$$

Fourier's brilliant idea[2] was to define this idea of projection for our space of functions as well! In specific, consider the following definition:

$$\begin{aligned}
\text{projection}\left(f(x), \sin(nx)\right) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx) dx \\
\text{projection}\left(f(x), \cos(nx)\right) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx \\
\text{projection}\left(f(x), \frac{1}{2})\right) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{2} dx.
\end{aligned}$$

Why are these definitions useful? Well, if you're Fourier, you made them because you've proved the following crazy/crazy-useful **orthogonality relations**:

$$\begin{aligned}
\int_{-\pi}^{\pi} \left(\frac{1}{2}\right) dx = \pi, \quad \int_{-\pi}^{\pi} \sin^2(nx) dx = \pi, \quad \int_{-\pi}^{\pi} \cos^2(nx) dx = \pi, \forall n \in \mathbb{N}. \\
\int_{-\pi}^{\pi} \sin(nx)\sin(mx) dx = 0, \quad \int_{-\pi}^{\pi} \cos(nx)\cos(mx) dx = 0, \forall n \neq m \in \mathbb{N}. \\
\int_{-\pi}^{\pi} \sin(nx)\cos(mx) dx = 0, \forall n, m \in \mathbb{N}. \\
\int_{-\pi}^{\pi} \sin(nx) \cdot \frac{1}{2} dx = 0, \quad \int_{-\pi}^{\pi} \cos(nx) \cdot \frac{1}{2} dx = 0, \forall n > 0 \in \mathbb{N}.
\end{aligned}$$

[1] Technically speaking, vector spaces only allow *finite* linear combinations of basis elements; so we're really working in something that's just vector-space-like. For our purposes, however, it has all of the vector space properties we're going to need, so it's a lot better for your intuition to just think of this as a vector space and not worry about the infinite-sum thing for now.

[2] Well, one of Fourier's many brilliant ideas. He had a lot.

What do these relations have to do with this strange definition of projection? Well, let's look at the $\sin(mx)$ projection onto a Fourier series $f(x) = \frac{c}{2} + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$:

$$\begin{aligned}
\operatorname{proj}(f(x), \sin(mx)) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(mx) dx \\
=& \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{c}{2} \cdot \sin(mx) + \sum_{n=1}^{\infty} (a_n \sin(nx) \sin(mx) + b_n \cos(nx) \sin(mx) dx) \\
=& \frac{1}{\pi} \left(\int_{-\pi}^{\pi} \frac{c}{2} \cdot \sin(mx) dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \sin(nx) \sin(mx) + b_n \cos(nx) \sin(mx)) dx \right)
\end{aligned}$$

But the orthogonality relations tell us that all of these individual integrals of the $\sin(nx)\sin(mx)$, $\cos(nx)\cos(mx), \sin(mx)/2$ terms are all 0, while the $\sin^2(mx)$ term has integral π. So, in specific, we can calculate this crazy thing, and see that it's just

$$\frac{1}{\pi} (0 + a_m \cdot \pi) = a_m.$$

In other words: projection works! I.e. if we have a Fourier series $f(x) = \frac{c}{2} + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$, we have

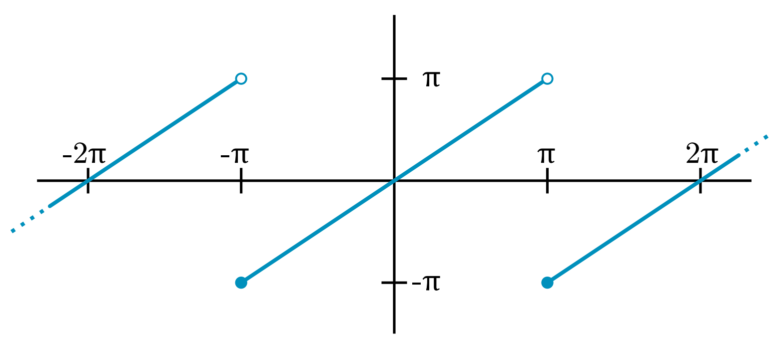
$$\begin{aligned}
&\operatorname{projection}(f(x), \sin(nx)) = a_n, \\
&\operatorname{projection}(f(x), \cos(nx)) = b_m, \\
&\operatorname{projection}\left(f(x), \frac{1}{2})\right) = c.
\end{aligned}$$

So we can turn functions into Fourier series!

1.3 How to Find a Fourier Series: An Example

To illustrate how this works in practice, consider the following example:

Example 1.3. Find the Fourier series of the **sawtooth wave** $s(x)$:



$$s(x) = \begin{cases} x, & x \in [-\pi, \pi), \\ s(x \pm 2\pi), & \forall x \in \mathbb{R}. \end{cases}$$

Solution: We proceed via the projection method we developed above:

$$\begin{aligned}
\text{(constant term)} &= \text{projection}\left(s(x), \frac{1}{2}\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cdot\frac{1}{2}dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\frac{1}{2}dx \\
&= \frac{1}{\pi}\left(\frac{x^2}{4}\right)\Big|_{-\pi}^{\pi} \\
&= 0. \\
\text{(}\sin(nx)\text{ term)} =& \text{projection}\left(s(x), \sin(nx)\right) \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cdot\sin(nx)dx \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)dx \\
=& \frac{1}{\pi}\left(x\cdot\frac{-\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx\right), \\
& \qquad \text{[via integration by parts with } u = x, dv = \sin(nx).\text{]} \\
=& \frac{1}{\pi}\left(\frac{\pi\cos(n\pi) + \pi\cos(-n\pi)}{n} + 0\right) \\
=& \frac{2\cos(n\pi)}{n}, \quad \text{[because cos is even.]} \\
=& \frac{2(-1)^{n+1}}{n}. \\
\text{(}\cos(nx)\text{ term)} =& \text{projection}\left(s(x), \cos(nx)\right) \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cdot\cos(nx)dx \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)dx \\
=& \frac{1}{\pi}\left(x\cdot\frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nx)}{n}dx\right), \\
& \qquad \text{[via integration by parts with } u = x, dv = \cos(nx).\text{]} \\
=& \frac{1}{\pi}\left(\frac{\pi\sin(n\pi) + \pi\sin(-n\pi)}{n} + 0\right) \\
=& 0.
\end{aligned}$$

Therefore, we've proven that the Fourier series for our sawtooth wave $s(x)$ is

$$\sum_{n=1}^{\infty}\frac{2\cdot(-1)n+1\cdot\sin(nx)}{n}.$$

2 Complex Power Series

In Ma1a, we often ran into the following question: "Given some polynomial $P(x)$, what are its roots?" Depending on the polynomial, we had several techniques for finding these roots (Rolle's theorem, quadratic/cubic formulas, factorization;) however, we at times would encounter polynomials that have no roots at all, like

$$x^2 + 1.$$

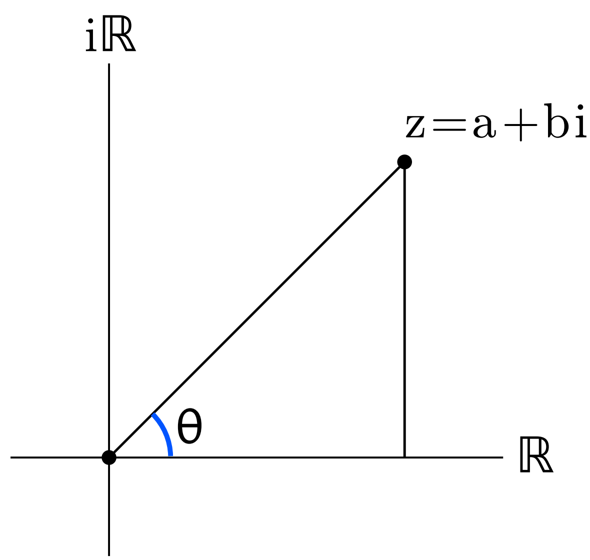
Yet, despite the observation that this polynomial's graph never crossed the x-axis, we *could* use the quadratic formula to find that this polynomial had the "formal" roots

$$\frac{-0 \pm \sqrt{-4}}{2} = \pm\sqrt{-1}.$$

The number $\sqrt{-1}$, unfortunately, isn't a real number (because $x^2 \geq 0$ for any real x, as we proved last quarter) – so we had that this polynomial has no roots over $\mathbb{R}$. This was a rather frustrating block to run into; often, we like to factor polynomials entirely into their roots, and it would be quite nice if we could always do so, as opposed to having to worry about irreducible functions like $x^2 + 1$.

Motivated by this, we can create the **complex numbers** by just throwing $\sqrt{-1}$ into the real numbers. Formally, we define the set of complex numbers, $\mathbb{C}$, as the set of all numbers $\{a + bi : a, b \in \mathbb{R}\}$, where $i = \sqrt{-1}$.

Graphically, we can visualize the complex numbers as a plane, where we identify one axis with the real line $\mathbb{R}$, the other axis with the imaginary-real line $i\mathbb{R}$, and map the point $a + bi$ to (a, b):



Two useful concepts when working in the complex plane are the ideas of **norm** and **conjugate**:

Definition 2.1. If $z = x + iy$ is a complex number, then we define $|z|$, the **norm** of z, to be the distance from z to the origin in our graphical representation; i.e. $|z| = \sqrt{x^2 + y^2}$.

As well, we define the **conjugate** of $z = x + iy$ to be the complex number $\overline{z} = x - iy$. Notice that $|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}$.

In the real line, recall that we had $|x \cdot y| = |x| \cdot |y|$; this still holds in the complex plane! In particular, we have $|w \cdot z| = |w| \cdot |z|$, for any pair of complex numbers w, z. (If you don't believe this, prove it! – it's not a difficult exercise to check.)

So: we have this set, $\mathbb{C}$, that looks like the real numbers with i thrown in. Do we have any way of extending any of the functions we know and like on $\mathbb{R}$, like $\sin(x), \cos(x), e^x$ to the complex plane?

At first glance, it doesn't seem likely: i.e. what should we say $\sin(i)$ is? Is cos a periodic function when we add multiples of $2\pi i$ to its input? Initially, these questions seem unanswerable; so (as mathematicians often do when faced with difficult questions) let's try something easier instead!

In other words, let's look at **polynomials**. These functions are much easier to extend to $\mathbb{C}$: i.e. if we have a polynomial on the real line

$$f(x) = 2x^3 - 3x + 1,$$

the natural way to extend this to the complex line is just to replace the x's with z's: i.e.

$$f(z) = 2z^3 - 3z + 1.$$

This gives you a well-defined function on the complex numbers (i.e. you put a complex number in and you get a complex number out,) such that if you restrict your inputs to the real line $x + i \cdot 0$ in the complex numbers, you get the same outputs as the real-valued polynomial.

In other words, we know how to work with polynomials. Does this help us work with more general functions? As we've seen over the last two weeks, the answer here is yes! More specifically, the answer here is to use **power series**. Specifically, over the last week, we showed that

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots, \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots, \text{and} \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots
\end{aligned}$$

for all real values of x. Therefore, we can choose to **define**

$$\begin{aligned}
\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots, \\
\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots, \text{and} \\
e^z &= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots,
\end{aligned}$$

for all $z \in \mathbb{C}$. This extension has the same properties as the one we chose for polynomials: it gives a nice, consistent definition of each of these functions over all of $\mathbb{C}$, that agrees with the definitions they already had on the real line $\mathbb{R}$.

The only issue with these extensions is that we're still not entirely quite sure what they mean. I.e.: what **is** $\sin(i)$, apart from some strange infinite power series? Where does the point e^z lie on the complex plane?

To answer these questions, let's look at e^z first, as it's arguably the easiest of the three (its terms don't do the strange alternating-thing, and behave well under most algebraic manipulations.) In particular, write $z = x + iy$: then we have

$$e^z = e^{x+iy} = e^x \cdot e^{iy},$$

where e^x is just the real-valued function we already understand. So, it suffices to understand e^{iy}, which we study here:

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \frac{(iy)^7}{7!} + \frac{(iy)^8}{8!} + \dots$$

If we use the fact that $i^2 = -1$, we can see that powers of i follow the form

$$i, -1, -i, 1, i, -1, -i, 1, \dots$$

and therefore that

$$e^{iy} = 1 + iy - \frac{y^2}{2} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \frac{y^6}{6!} - i\frac{y^7}{7!} + \frac{y^8}{8!} + \dots$$

If we split this into its real and imaginary parts, we can see that

$$e^{iy} = \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} \dots\right).$$

But wait! We've seen those two series before: they're just the series for $\sin(y)$ and $\cos(y)$! In other words, we've just shown that

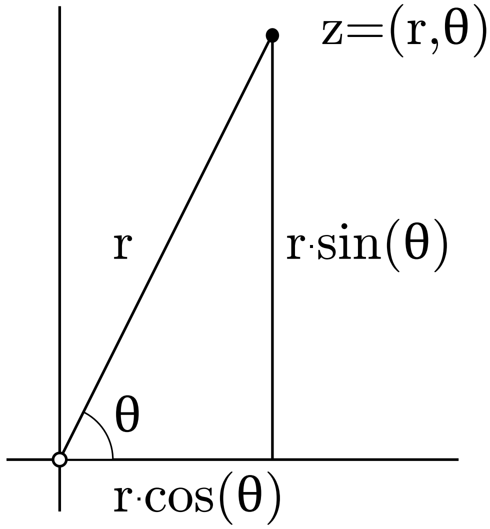
$$e^{iy} = \cos(y) + i\sin(y).$$

One famous special case of this formula is when $y = \pi$, in which case we have $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$, or

$$e^{i\pi} + 1 = 0.$$

Which is *amazing*. In one short equation, we've discovered a fundamental relation that connects five of the most fundamental mathematical constants, in a way that – without this language of power series and complex numbers – would be unfathomable to understand. Without power series, the fact that a constant related to the area of a circle (π), the square root of negative 1, the concept of exponential growth (e) and the multiplicative identity (1) can be combined to get the additive identity (0) would just seem absurd; yet, with them, we can see that this relation was inevitable from the very definitions we started from.

But that's not all! This formula (Euler's formula) isn't just useful for discovering deep fundamental relations between mathematical constants: it also gives you a way to visualize the complex plane! In specific, recall the concept of **polar coördinates**, which assigned to each nonzero point z in the plane a value $r \in \mathbb{R}^+$, denoting the distance from this point to the origin, and an angle $\theta \in [0, 2\pi)$, denoting the angle made between the positive x-axis and the line connecting z to the origin:



With this definition made, notice that any point with polar coördinates (r, θ) can be written in the plane as $(r\cos(\theta), r\sin(\theta))$. This tells us that any point with polar coördinates (r, θ) in the complex plane, specifically, can be written as $r(\cos(\theta) + i\sin(\theta))$; i.e. as $re^{i\theta}$.

This gives us what we were originally looking for: a way to visually interpret e^{x+iy}! In specific, we've shown that e^{x+iy} is just the point in the complex plane with polar coördinates (e^x, y).