MATH 8, SECTION 1 - FINAL REVIEW NOTES (DEFINITIONS)

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Abstract. These are the definition-half of the notes from Monday, Dec. 6rd's final review; here, we summarize all of the major results we've discussed this quarter.

1. Bestiary of Functions

For convenience's sake, we list the definitions, integrals, derivatives, and key values of several functions here.

Name	Domain	Derivative	Integral	Key Values
$\ln(x)$	$(0, \infty)$	$1/x$	$x \cdot \ln(x) - x + C$	$\ln(1) = 0$, $\ln(e) = 1$
e^x	$\mathbb{R}$	e^x	$e^x + C$	$e^0 = 1$, $e^1 = e$
$\sin(x)$	$\mathbb{R}$	$\cos(x)$	$-\cos(x) + C$	$\sin(0) = 0$, $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\sin\left(\frac{\pi}{2}\right) = 1$
$\cos(x)$	$\mathbb{R}$	$-\sin(x)$	$\sin(x) + C$	$\cos(0) = 1$, $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{\pi}{2}\right) = 0$
$\tan(x)$	$x \neq \frac{2k+1}{2}\pi$	$\sec^2(x)$	$\ln\|\sec(x)\| + C$	$\tan(0) = 0$, $\tan\left(\frac{\pi}{4}\right) = 1$
$\sec(x)$	$x \neq \frac{2k+1}{2}\pi$	$\sec(x)\tan(x)$	$\ln\|\sec(x) + \tan(x)\| + C$	$\sec(0) = 1$, $\sec\left(\frac{\pi}{4}\right) = \frac{2\sqrt{2}}{2}$
$\csc(x)$	$x \neq k\pi$	$-\csc(x)\cot(x)$	$\ln\|\csc(x) - \cot(x)\| + C$	$\csc\left(\frac{\pi}{4}\right) = \frac{2\sqrt{2}}{2}$, $\sec\left(\frac{\pi}{2}\right) = 1$
$\cot(x)$	$x \neq k\pi$	$-\csc^2(x)$	$\ln\|\sin(x)\| + C$	$\cot\left(\frac{\pi}{4}\right) = 1$, $\cot\left(\frac{\pi}{2}\right) = 0$
$\arcsin(x)$	$(-1, 1)$	$\frac{1}{\sqrt{1-x^2}}$	$x\arcsin(x) + \sqrt{1-x^2} + C$	$\arcsin(0) = 0$, $\arcsin(1) = \frac{\pi}{2}$
$\arccos(x)$	$(-1, 1)$	$-\frac{1}{\sqrt{1-x^2}}$	$x\arccos(x) - \sqrt{1-x^2} + C$	$\arccos(0) = \frac{\pi}{2}$, $\arcsin(1) = 0$
$\arctan(x)$	$\mathbb{R}$	$\frac{1}{1+x^2}$	$x\arctan(x) - \frac{\ln(1+x^2)}{2} + C$	$\arctan(0) = 0$, $\arctan(1) = \frac{\pi}{2}$

2. Concepts and Theorems

In this section, we list several of the most important concepts we've studied this quarter:

2.1. Sequences and Series. [1]

2.1.1. *Basic Definitions.*

(1) **Sequence**: A **sequence** is just a collection $\{a_n\}_{n=1}^{\infty}$ of objects (usually numbers) indexed by $\mathbb{N}$.

(2) **Convergence**: A sequence $\{a_n\}_{n=1}^{\infty}$ of numbers – either complex or real! – **converges** to some value λ (i.e. $\lim_{n\to\infty} a_n = \lambda$) if

$$(\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

(3) **Series**: Given any sequence $\{a_n\}_{n=1}^{\infty}$ of numbers, we can look at the limit of the partial sums of this sequence $\left\{\sum_{n=1}^{N} a_n\right\}_{N=1}^{\infty}$. If this sequence of partial sums converges, we call the limit of these partial sums the **series** corresponding to the sequence $\{a_n\}_{n=1}^{\infty}$.

(4) **Absolute convergence**: A series $\sum_{n=1}^{\infty} a_n$ converges **absolutely** iff the series $\sum_{n=1}^{\infty} |a_n|$ converges; it converges **conditionally** iff the series $\sum_{n=1}^{\infty} a_n$ converges and the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ diverges.

(5) **Power series**: A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n x^n$. If the a_n's are all real numbers, then this is called a real power series; if they are complex numbers, then this is a complex power series.

(6) **Radius of convergence**: For a given power series (either real or complex) $\sum_{n=0}^{\infty} a_n x^n$, we say that the **radius of convergence** of this series is some value $R \in \mathbb{R}$ such that
- if x is a number such that $|x| < R$, $\sum_{n=0}^{\infty} a_n x^n$ converges, and
- if x is a number such that $|x| > R$, $\sum_{n=0}^{\infty} a_n x^n$ diverges.

Note that if $|x| = R$, we have no idea what happens: our sequence can either converge or diverge.

(7) **Pointwise convergence**: A sequence of functions $f_n(x)$ is said to **pointwise converge** to some function $F(x)$ on some set X iff for any $x \in X$, $\lim_{n\to\infty} f_n(x) = F(x)$.

(8) **Uniform convergence**: A sequence of functions $f_n(x)$ is said to **uniformly converge** to some function $F(x)$ on some set X iff for any $\epsilon > 0$, there is some N such that for all $n > N$ and $x \in X$, we have that $|f_n(x) - F(x)| < \epsilon$.

2.1.2. *Theorems and Tools.*

For sequences:

(1) **Monotone and bounded sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges.

(2) **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l.

For real-valued series:

[1] Relevant lectures: Friday, wk. 2, Monday, wk. 3, Monday, wk. 3, Wednesday, wk. 3, Friday, wk. 3, Friday, wk. 8, Friday, wk. 10.

(1) **Comparison test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then the following statement is true:
$$\left(\sum_{n=1}^{\infty} b_n \text{ converges} \right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges} \right).$$
(2) **Limit comparison test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences of positive numbers such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$, then the following statement is true:
$$\left(\sum_{n=1}^{\infty} b_n \text{ converges} \right) \Leftrightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges} \right).$$
(3) **Alternating series test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that
- $\lim_{n\to\infty} a_n = 0$ monotonically, and
- the a_n's alternate in sign, then

the series $\sum_{n=1}^{\infty} a_n$ converges.

(4) **Ratio test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$
then we have the following three possibilities:
- If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then we have no idea; it could either converge or diverge.

(5) **Root test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that
$$\lim_{n\to\infty} \sqrt[n]{a_n} = r,$$
then we have the following three possibilities:
- If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then we have no idea; it could either converge or diverge.

(6) **Integral test**: If $\{a_n\}_{n=k}^{\infty}$ is a sequence of numbers and $f(x)$ is a positive monotone-decreasing function such that $f(n) = a_n$, then the series $\sum_{n=k}^{\infty} a_n$ converges iff the integral $\int_k^{\infty} f(x)dx$ exists.

(7) **Absolute convergence and convergence**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

(8) **Vanishing criterion**: If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

For power series:

(1) **Radius of convergence, real**: Every real-valued power series $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence R.

(2) **Radius of convergence, complex**: Every complex-valued power series $\sum_{n=0}^{\infty} \alpha_n z^n$ has a radius of convergence R.

(3) **Radius of convergence, agreement**: Suppose that $\sum_{n=0}^{\infty} \alpha_n z^n$ is a complex-valued power series where all of the α_n's are real numbers. Then the radius of convergence of the complex power series $\sum_{n=0}^{\infty} \alpha_n z^n$ is the same as that of the real power series formed from the same coefficients, $\sum_{n=0}^{\infty} \alpha_n x^n$.

(4) **Power series and uniform convergence**: Pick a real power series $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, if our power series converges at some value x, then it

converges uniformly to $F(x)$ on any interval $[-b, b]$, for any $b < |x|$. Furthermore, the power series $G(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges uniformly on $[-b, b]$, and $G(x) = F'(x)$ on $[-b, b]$.

2.2. **Limits and Continuity.** [2]

2.2.1. *Basic Definitions.*

(1) **Limits**: If $f : X \to Y$ is a function between two subsets X, Y of $\mathbb{R}$, we say that

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

(2) **Continuity**: A function $f : X \to Y$ is said to be **continuous** at some point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a).$$

2.2.2. *Theorems and Tools.*

(1) **A blueprint for $\epsilon - \delta$ proofs of limits**: In class, we developed the following "blueprint" that describes a general method for proving that $\lim_{x \to a} f(x) = L$ via an $\epsilon - \delta$ argument. We review this below:
   (a) First, examine the quantity

   $$|f(x) - L|.$$

   Specifically, try to find a simple upper bound for this quantity that depends only on $|x - a|$, and goes to 0 as x goes to a – something like $|x - a| \cdot (\text{constants})$, or $|x - a|^3 \cdot (\text{bounded functions, like } \sin(x))$.
   (b) Using this simple upper bound, for any $\epsilon > 0$, choose a value of δ such that whenever $|x - a| < \delta$, your simple upper bound $|x - a| \cdot (\text{constants})$ is $< \epsilon$. Often, you'll define δ to be $\epsilon/(\text{constants})$, or somesuch thing.
   (c) Plug in the definition of the limit: for any $\epsilon > 0$, we've found a δ such that whenever $|x - a| < \delta$, we have

   $$|f(x) - L| < (\text{simple upper bound depending on } |x - a|) < \epsilon.$$

   If you've done all of this, you've proved $\lim_{x \to a} f(x) = L$.

(2) **Squeeze theorem**: If f, g, h are functions defined on some interval $I \setminus \{a\}$ such that

$$f(x) \le g(x) \le h(x), \forall x \in I \setminus \{a\}, \text{ and}$$
$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x),$$

then $\lim_{x \to a} g(x)$ exists, and is equal to the other two limits $\lim_{x \to a} f(x), \lim_{x \to a} h(x)$.

(3) **Intermediate value theorem**: If f is a continuous function on $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$ at least once.

(4) **L'Hôpital's rule**: If $f(x)$ and $g(x)$ are a pair of differentiable functions such that either
   - $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, or

[2] Relevant lectures: Monday, wk. 4, Wednesday, wk. 4, Friday, wk. 4.

- $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, whenever the second limit exists.

2.3. Differentiation. [3]

2.3.1. *Basic Definitions.*

(1) **The derivative**: Given a function f defined on some neighborhood $(a-\delta, a+\delta)$, we say that f is **differentiable** at a iff the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a}$$

exists. If it does, denote this limit as $f'(a)$.

(2) **Critical points**: For a function f defined on some set X, the **critical points** of f on X are all of the points in x where either
- $f'(x) = 0$, or
- $f'(x)$ doesn't exist.

(3) **Maxima and minima**: For a function f defined on some interval (a, b), a point $x \in (a, b)$ is called a **local maxima** iff there is some small neighborhood $(x-\delta, x+\delta)$ in which $f(x) > f(t), \forall t \in (x-\delta, x+\delta)$.

Similarly, a point $x \in (a, b)$ is called a **local minima** iff there is some small neighborhood $(x-\delta, x+\delta)$ in which $f(x) < f(t), \forall t \in (x-\delta, x+\delta)$.

2.3.2. *Theorems and Tools.*

(1) **Product rule**: For f, g a pair of functions differentiable at a,

$$(f(x)\cdot g(x))'\Big|_a = f'(a)\cdot g(a) + g'(a)\cdot f(a).$$

(2) **Quotient rule**: For f, g a pair of functions differentiable at a, $g(a) \neq 0$, we have

$$\left(\frac{f(x)}{g(x)}\right)'\Bigg|_a = \frac{f'(a)\cdot g(a) - g'(a)\cdot f(a)}{(g(a))^2}$$

(3) **Chain rule**: For f a function differentiable at $g(a)$ and g a function differentiable at a,

$$(f(g(x)))'\Big|_a = f'(g(a))\cdot g'(a).$$

(4) **Critical points and local minima/maxima**:If x is a local minima or maxima for some function f, x is a critical point of f.

(5) **Critical points and global minima/maxima**: If f is a continuous function on some interval $[a, b]$, then f takes on its minima and maxima over this entire region. Furthermore, f takes on these minimum and maximum values at either the critical points of f or at the endpoints $\{a, b\}$.

[3] Relevant lectures: Monday, wk. 5, Wednesday, wk. 5, Wednesday, wk. 8.

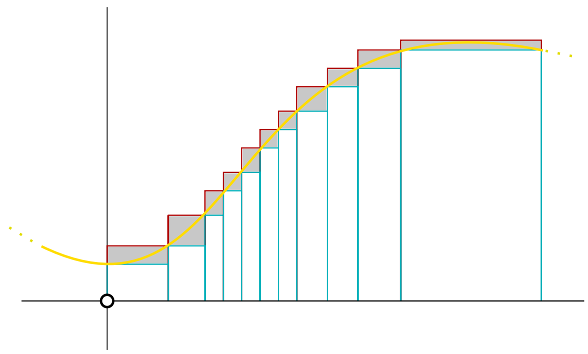
2.4. **Integration.** [4]

2.4.1. *Basic Definitions.*

(1) **The integral**: A function f is **integrable** on the interval $[a, b]$ if and only if the following holds:
- For any $\epsilon > 0$,
- there is a partition $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of the interval $[a, b]$ such that

$$\left(\sum_{i=1}^{n} \sup_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i) - \sum_{i=1}^{n} \inf_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i) \right) < \epsilon.$$

Pictorially, this is just saying that the area of the blue rectangles approaches the area of the red rectangles in the picture below:



Because of this picture, we often say that the integral of a function on some interval $[a, b]$ is the **area** beneath its curve from $x = a$ to $x = b$.

(2) **Negligible**: A set $X \subset \mathbb{R}$ is called **negligible** if for any $\epsilon > 0$, there is some collection $\{I_n\}_{n=1}^{\infty}$ of closed intervals of positive length, such that

$$(1) \quad \bigcup_{n=1}^{\infty} I_n, \text{ the union of all of these intervals, } \supseteq X.$$

$$(2) \quad \sum_{n=1}^{\infty} \text{length}(I_n) \leq \epsilon.$$

Any countable or finite set is negligible.

(3) **Primitive**: A function $f(x)$ has $\varphi(x)$ as its **primitive** on some interval $[a, b]$ iff $\varphi'(x) = f(x)$ on all of $[a, b]$.

2.4.2. *Theorems and Tools.*

(1) **Functions with a negligible set of discontinuities**: If $f(x)$ is a bounded function on the interval $[a, b]$, and the collection of $f(x)$'s discontinuities on $[a, b]$ is a negligible set, then the integral

$$\int_a^b f(x)dx$$

[4]Relevant lectures: Monday, wk. 6, Wednesday, wk. 6, Friday, wk. 6, Monday, wk. 7, Wednesday, wk. 7, Friday, wk. 7.

exists.

(2) **The first fundamental theorem of calculus**: Let $[a, b]$ be some interval. Suppose that f is a bounded and integrable function over the interval $[a, x]$, for any $x \in [a, b]$. Then the function

$$A(x) := \int_a^x f(t)dt$$

exists for all $x \in [a, b]$. Furthermore, if $f(x)$ is continuous, the derivative of this function, $A'(x)$, is equal to $f(x)$.

(3) **The second fundamental theorem of calculus**: Let $[a, b]$ be some interval. Suppose that $f(x)$ is a function that has $\varphi(x)$ as its primitive on $[a, b]$; as well, suppose that $f(x)$ is bounded and integrable on $[a, b]$. Then, we have that

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a).$$

(4) **Integration by parts**: If f, g are a pair of C^1 functions on $[a, b]$ – i.e they have continuous derivatives on $[a, b]$ – then we have

$$\int_a^b f(x)g'(x) = f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx.$$

(5) **Integration by substitution**: If f is a continuous function on $g([a, b])$ and g is a C^1 functions on $[a, b]$, then we have

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx.$$

2.5. The Exponential and the Logarithm. [5]

2.5.1. *Basic Definitions.*

(1) **The natural logarithm**: For $x \in (0, \infty)$, we define $\ln(x) = \int_1^x \frac{1}{x}dx$.

(2) **The exponential function**: For any $x \in \mathbb{R}$, we define $\exp(x)$ as the unique value y such that $\ln(y) = x$. In other words, $\exp(x)$ is the inverse function to $\ln(x)$.

2.5.2. *Theorems and Tools.*

(1) $\exp$ **and** e: $\exp(x) = e^x$, where e is the mathematical constant equal to $\exp(1) = 2.71\ldots$.

(2) **A third definition of** e^x:For any x, we have

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

[5]Relevant lectures: Friday, wk. 3, Wednesday, wk. 8, Monday, wk. 9.

2.6. Taylor Series. [6]

2.6.1. *Basic Definitions.*

(1) **Taylor polynomials**: For $f(x)$ a n-times differentiable function at a, we define the **n-th Taylor polynomial of** $f(x)$ **around** a as the following degree-n polynomial:

$$T_n(f(x), a) := \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n.$$

(2) **Remainder function**: For $f(x)$ a n-times differentiable function at a, we define the **n-th order remainder function of** $f(x)$ **around** a as follows:

$$R_n(f(x), a) = f(x) - T_n(f(x), a).$$

(3) **Taylor series**: For $f(x)$ an infinitely-differentiable function at a, we define its Taylor series as the following power series around a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n.$$

If $\lim_{n\to\infty} R_n(f(x), a) = 0$ at some value of x, then this Taylor series is in fact equal to $f(x)$ at x.

(4) **Agreement up to order** n: A pair of functions $f(x), g(x)$ are said to **agree up to order** n at a iff

$$\lim_{x\to a} \frac{f(x) - g(x)}{(x-a)^n} = 0.$$

2.6.2. *Theorems and Tools.*

(1) **Taylor polynomials agree up to order** n **with their functions**: If $f(x)$ is a n-times differentiable function at a, then $T_n(f(x), a)$ agrees with f up to order n at a. Furthermore, $T_n(f(x), a)$ is the **only** polynomial with degree $\leq n$ that agrees with $f(x)$ up to order n at a.

(2) **Taylor's theorem**: If $f(x)$ is a $n+1$-times differentiable function on some open interval containing $[a, x]$, then there is a value c in the set (a, x) such that

$$R_n(f(x), a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

(3) **Several useful Taylor polynomials and series:** The following functions have the indicated Taylor polynomials and series, which converge on the indicated regions:

$T_{2n}(\cos(x), 0) = \sum_{k=0}^{n} (-1)^n \frac{x^{2n}}{(2n)!}$,	$\cos(x) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$,	$\forall x \in \mathbb{R}$
$T_{2n+1}(\sin(x), 0) = \sum_{k=0}^{n} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,	$\sin(x) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,	$\forall x \in \mathbb{R}$
$T_n(e^x, 0) = \sum_{k=0}^{n} \frac{x^k}{k!}$,	$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$,	$\forall x \in \mathbb{R}$
$T_n(\log(x+1), 0) = \sum_{k=1}^{n} (-1)^{n+1} \frac{x^n}{x}$,	$\log(x+1) = \sum_{k=1}^{\infty} (-1)^{n+1} \frac{x^n}{x}$,	$\forall -1 < x \leq 1$
$T_{2n+1}(\arctan(x), 0) = \sum_{n=0}^{n} \frac{(-1)^n}{2n+1} x^{2n+1}$,	$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$,	$\forall \lvert x\rvert < 1$
$T_n((x+y)^a, 0) = \sum_{k=0}^{n} \binom{a}{k} x^k y^{a-k}$,	$(x+y)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k y^{a-k}$,	$\forall \lvert x\rvert < \lvert y\rvert$
$T_n\left(\frac{1}{1-x}, 0\right) = \sum_{k=0}^{n} x^k$,	$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$,	$\forall \lvert x\rvert < 1$

[6] Relevant lectures: Monday, wk. 9, Wednesday, wk. 9, Friday, wk. 10.

2.7. **The Complex Plane.** [7]

2.7.1. *Basic Definitions.*

(1) **The complex plane**: The **complex plane** is the collection of all points of the form $a+bi$, where a, b are real numbers and i is the **imaginary unit**, a symbol with the property that $i^2 = -1$.

(2) **Defining functions on** $\mathbb{C}$: We define $\sin(z), \cos(z)$, and e^z on the complex plane as follows:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots,$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots, \text{ and}$$
$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots.$$

2.7.2. *Theorems and Tools.*

(1) **Polar coördinates and** $\mathbb{C}$: If z is a point in the complex plane with polar coördinates (r, θ), then $z = re^{i\theta}$.

(2) **The fundamental theorem of algebra**: Any nonconstant polynomial with coefficients in $\mathbb{C}$ has at least one root.

[7] Relevant lectures: Monday, wk. 10, Friday, wk. 10.