MATH 8, SECTION 1 - FINAL REVIEW NOTES (EXAMPLES)

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Abstract. These are the other half of the notes from Monday, Dec. 6rd's final review; here, we study examples of the major concepts encountered this quarter.

1. Problem 1: ϵ-δ proofs and complex polar coördinates

Question 1.1.

(1) *Prove that $f(x) = x^3 - 1$ is a continuous function on all of $\mathbb{R}$.*
(2) *What are this function's roots over $\mathbb{C}$?*
(3) *What are this function's global minima and maxima over the interval $[-1, 1]$?*

Proof. (1): To prove this, let's try using the "Blueprint for $\epsilon - \delta$ proofs" in the notes/final review handout. Specifically, let's do the following:

(1) First, let's look at $|f(x)-f(a)|$, and try to create a simple bound depending only on $|x-a|$ and some constants.

$$|f(x) - f(a)| = |x^3 - 1 - a^3 + 1| = |x^3 - a^3| = |x - a| \cdot |x^2 + xa + a^2|.$$

If x is within, say, 1 of a, we know that we can bound this quantity $|x^2 + xa + a^2|$ as follows:

$$|x^2 + xa + a^2| \leq |(a + 1)^2 + a(a + 1) + a^2| \leq 3(a + 1)^2,$$

which is a constant! Therefore, whenever x is within 1 of a, we have the following simple bound:

$$|f(x) - f(a)| \leq |x - a| \cdot (3(a + 1)^2).$$

(2) Now that we have this nice constant bound, we want to pick δ such that whenever $|x-a| < \delta$, $|f(x) - f(a)| < \epsilon$. To do this, we simply want to pick δ such that
- $\delta < 1$, so that x is always forced to be within 1 of a, and we have our nice constant bound, and
- $\delta < \frac{\epsilon}{3(a+1)^2}$, because this means that

$$|f(x) - f(a)| \leq |x - a| \cdot (3(a + 1)^2) < \frac{\epsilon}{3(a + 1)^2} \cdot 3(a + 1)^2 = \epsilon$$

So: let $\delta < \min\left(1, \frac{\epsilon}{3(a+1)^2}\right)$.

Then δ is smaller than both 1 and $\frac{\epsilon}{3(a+1)^2}$, and so both of our above statements hold! In particular, for any epsilon, this choice of δ forces

$$|f(x) - f(a) < \epsilon,$$

which is exactly what we want to do in an $\epsilon - \delta$ proof to show continuity.

(2): Finding this function's roots over $\mathbb{C}$ is equivalent to finding all of the values of z such that

$$1 = z^3.$$

To do this: first, remember that we can write any nonzero point in $\mathbb{C}$ with polar coördinates (r, θ) uniquely in the form $re^{i\theta}$, where $r \in (0, \infty)$ and $\theta \in [0, 2\pi]$. Then, we're just looking for all of the values r, θ such that

$$1 = r^3 e^{3i\theta}.$$

Notice that if the above equation holds, then we have that

$$1 = \left|r^3 e^{3i\theta}\right| = \left|r^3\right| \cdot \left|e^{3i\theta}\right|.$$

However, if we use the formula $e^{ix} = \cos(x) + i\sin(x)$ and the definition $|a + bi| = \sqrt{a^2 + b^2}$, we can see that

$$\begin{aligned}\left|e^{3i\theta}\right| &= |\cos(3\theta) + i\sin(3\theta)) \\ &= \sqrt{\cos^2(3\theta) + \sin^2(3\theta)} \\ &= \sqrt{1} \\ &= 1.\end{aligned}$$

Therefore, we in fact have that $r^3 = 1$; i.e. $r = 1$! All we have to do now is then solve for θ.

We do this in a similar way: if we have $e^{3i\theta} = 1$, by using $e^{ix} = \cos(x) + i\sin(x)$ again, we must have that

$$\begin{aligned} & 1 = \cos(3\theta) + i\sin(3\theta) \\ \Rightarrow \quad & \cos(3\theta) = 1, \text{ and } \sin(3\theta) = 0.\end{aligned}$$

The three values $\theta = 0, 2\pi/3, 4\pi/3$ are solutions to the above, and therefore correspond to the three roots $1, e^{2i\pi/3}, e^{4i\pi/3}$ of $f(z) = z^3 - 1$; by the fundamental theorem of calculus, we know that there are only three roots, and thus that we've found them all.

(3): Finally, we can find the minima and maxima of this (now real-valued, again) function on $[-1, 1]$ by simply taking its derivative. As $f'(x) = 3x^2$ has its only 0 at 0, we know (by the extremal value theorem) that the only points we have to check for extrema are $x = -1, 0$, and 1. Because $f(-1) = -2, f(0) = -1$, and $f(1) = 0$, we know that its global maxima on this interval is 0 and its global minima is -2. $\square$

2. Problem 2: Taylor polynomials and series

Question 2.1.

(1) *Find* $T_{2n}\left(e^{x^2}, 0\right)$, *and the associated Taylor series for* e^{x^2}.
(2) *Where does this Taylor series converge? Where does it converge absolutely? Where does it converge uniformly?*
(3) *Approximate* $\int_{-1/2}^{1/2} e^{x^2} dx$ *with an error of about* $\pm .1$.

Proof. (1): We proceed in a similar fashion to Wednesday, week 9's notes. First, recall that we can always write e^t, for any value of t, as the following power series:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

So, in specific, if we let $t = x^2$, we have that

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

This motivates us to make the following claim:

Claim 2.2.

$$T_{2n}\left(e^{x^2}, 0\right) = \sum_{k=0}^{n} \frac{x^{2k}}{k!}.$$

Proof. By a theorem from class/on page 8 of the final review handout, we know that this is true iff $\sum_{k=0}^{n} \frac{x^{2k}}{k!}$ and e^{x^2} agree up to order $2n$ at 0. (This is because the $2n$-th Taylor polynomial of a function is the unique polynomial of degree $\leq 2n$ that agrees with its function up to order $2n$.) Therefore, to prove our claim, it suffices to show that

$$\lim_{x \to 0} \frac{e^{x^2} - \sum_{k=0}^{n} \frac{x^{2k}}{k!}}{x^{2n}} = 0.$$

To see this: simply make the substitution $y = x^2$. Then the left-hand-side above becomes

$$\lim_{y \to 0} \frac{e^{y} - \sum_{k=0}^{n} \frac{y^{k}}{k!}}{y^{n}},$$

which we know is 0 because $T_n(e^y, 0) = \frac{y^k}{k!}$, and therefore these two functions agree up to order n at 0. Therefore, we've proven that

$$T_{2n}\left(e^{x^2}, 0\right) = \sum_{k=0}^{n} \frac{x^{2k}}{k!},$$

and furthermore that e^{x^2}'s Taylor series is precisely

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

□

(2): If we apply the ratio test, we can see that for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{|x|^{2n+2}/(n+1)!}{|x|^{2n}} n! = \lim_{n\to\infty} \frac{|x|^2}{n+1} = 0,$$

and therefore for any value of x, the series

$$\sum_{n=0}^{\infty} \frac{|x|^{2n}}{n!}$$

converges: i.e. that

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

converges absolutely. Therefore, because absolute convergence implies convergence, we know that this power series converges on all of $\mathbb{R}$.

Furthermore, we know from a theorem from class / from page 2 of the final review handout that (for a power series $F(x) = \sum a_n x^n$,) "if our power series converges at some value x, then it converges uniformly to $F(x)$ on any interval $[-b, b]$, for any $b < |x|$." Therefore, we have that our Taylor series converges uniformly to e^{x^2} on any interval $[-b, b]$, for any $b \in \mathbb{R}^+$.

(3): Finally, to approximate $\int_{-1/2}^{1/2} e^{x^2} dx$, we write e^{x^2} as the sum of its second-order Taylor polynomial and second-order error term:

$$\begin{aligned} e^{x^2} &= T_2\left(e^{x^2}, 0\right) + R_2\left(e^{x^2}, 0\right) \\ &= 1 + x^2 + R_2\left(e^{x^2}, 0\right). \end{aligned}$$

By Taylor's theorem, we know that for x in the interval $[0, 1/2]$, we have

$$\begin{aligned} R_2\left(e^{x^2}, 0\right) &= \frac{\frac{\partial^3}{\partial x^3}\left(e^{x^2}\right)\Big|_c}{3!} x^3 \\ &= \frac{\frac{\partial^2}{\partial x^2}\left(2xe^{x^2}\right)\Big|_c}{3!} x^3 \\ &= \frac{\frac{\partial}{\partial x}\left((2+4x^2)e^{x^2}\right)\Big|_c}{3!} x^3 \\ &= \frac{\left((12c + 8c^3)e^{c^2}\right)}{3!} x^3, \end{aligned}$$

for some $c \in (0, 1/2)$.

So: because $\frac{\partial^3}{\partial c^3}\left(e^{c^2}\right)$ is monotonically increasing, we know that we can find an upper bound on it from by plugging in $c = 1/2$, and a lower bound by plugging in

$c = 0$. Doing this gives us $0 \leq \frac{\partial^3}{\partial c^3}\left(e^{c^2}\right) \leq 14$, by doing a few quick/dirty estimates (i.e. $\sqrt[4]{e} < 2$, and evaluating the poly at $1/2 = 7$.)

Applying this to our remainder function tells us that

$$0 \leq R_2\left(e^{x^2}, 0\right) \leq \frac{14}{6}x^3,$$

for $x \in (0, 1/2)$. Consequently, because we can write the integral

$$\begin{aligned}\int_{-1/2}^{1/2} e^{x^2}dx &= 2 \cdot \int_0^{1/2} e^{x^2}dx \\ &= 2 \cdot \int_0^{1/2} 1 + x^2 + R_2\left(e^{x^2}, 0\right)dx,\end{aligned}$$

we can use the bounds that we've found for $R_2\left(e^{x^2}, 0\right)$ on the interval $[0, 1/2]$ to get bounds on this integral:

$$\begin{aligned}\int_{-1/2}^{1/2} e^{x^2}dx &= 2 \cdot \int_0^{1/2} 1 + x^2 + R_2\left(e^{x^2}, 0\right)dx \\ &\leq 2 \cdot \int_0^{1/2} 1 + x^2 + \frac{14}{6}x^3 dx \\ &= 2 \cdot \left(x + \frac{x^3}{3} + \frac{14}{24}x^4\right)\Big|_0^{1/2} \\ &= 2 \cdot \left(\frac{1}{2} + \frac{1}{24} + \frac{7}{192}\right) \\ &= \frac{13}{12} + \frac{14}{192}, \text{ and} \\ \int_{-1/2}^{1/2} e^{x^2}dx &= 2 \cdot \int_0^{1/2} 1 + x^2 + R_2\left(e^{x^2}, 0\right)dx \\ &\geq 2 \cdot \int_0^{1/2} 1 + x^2 + 0dx \\ &= 2 \cdot \left(x + \frac{x^3}{3}\right)\Big|_0^{1/2} \\ &= \frac{13}{12}.\end{aligned}$$

Thus, we've shown that this integral lies somewhere between $13/12$ and $13/12 + 14/192$, which is as accurate as we wanted.

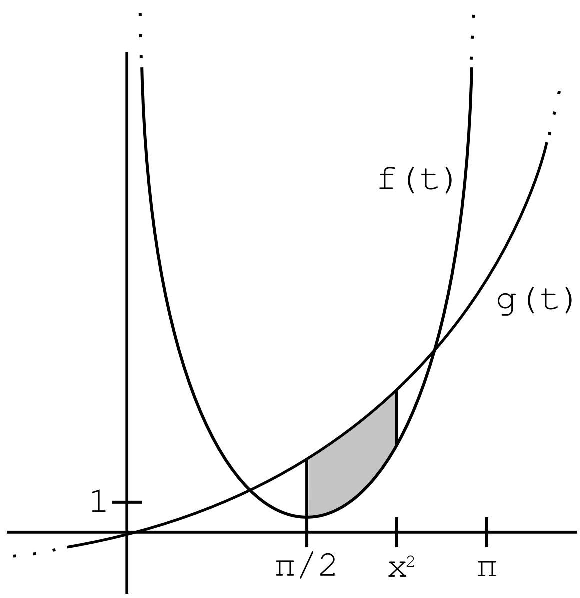
□

3. Problem 3: Integration techniques

Question 3.1.

(1) *Find the area bounded between the two curves* $f(t) = \frac{1}{\sin(t)}$ *and* $g(t) = te^t$ *from* $\pi/2$ *to* x^2*, where* $x^2 \in (\sqrt{\pi/2}, \sqrt{3\pi/4})$.
(2) *If* $F(x)$ *is the function that on input* x *returns the above area, what's* $F'(x)$*?*

Proof. (1): We start by graphing both of these functions, so that we can better understand the area we're trying to calculate:



As we can see in the picture above, te^t is always greater than $\frac{1}{\sin(t)}$ on the interval we're studying: therefore, the area bounded between the two curves is just the difference between the area bounded by $g(t)$ and the area bounded by $f(t)$: i.e.

$$\text{area} = \int_{\pi/2}^{x^2} te^t dt - \int_{\pi/2}^{x^2} \frac{1}{\sin(t)} dt.$$

To calculate the first integral, we proceed via integration by parts, setting

$$\begin{array}{ll} u = t & dv = e^t dt \\ du = dt & v = e^t. \end{array}$$

This tells us that

$$\begin{aligned} \int_{\pi/2}^{x^2} te^t dt &= te^t \Big|_{\pi/2}^{x^2} - \int_{\pi/2}^{x^2} e^t dt \\ &= \left(te^t - e^t\right) \Big|_{\pi/2}^{x^2} \\ &= \left(x^2 - 1\right) e^{x^2} - \frac{\pi^2 - 1}{4} e^{\pi^2/4}. \end{aligned}$$

To find the second integral, we first notice the following algebraic identity:

$$\begin{aligned}\frac{1}{\sin(t)} &= \frac{\sin(t)}{\sin^2(t)} \\ &= \frac{\sin(t)}{1-\cos^2(t)} \\ &= \frac{1}{2}\left(\frac{\sin(t)}{1+\cos(t)} + \frac{\sin(t)}{1-\cos(t)}\right).\end{aligned}$$

(We did something very similar on Friday, wk. 7, to calculate the integral of $\sec(x)$.) With this identity, we can then use integration by substitution (with the two substitutions $u = 1 \pm \cos(x), du = \mp \sin(x)$) to find the second integral:

$$\begin{aligned}\int_{\pi/2}^{x^2} \frac{1}{\sin(t)} dt &= \int_{\pi/2}^{x^2} \frac{1}{2}\left(\frac{\sin(t)}{1+\cos(t)} + \frac{\sin(t)}{1-\cos(t)}\right) dt \\ &= \frac{1}{2} \cdot \int_{\pi/2}^{x^2} \frac{\sin(t)}{1+\cos(t)} dt + \frac{1}{2} \cdot \int_{\pi/2}^{x^2} \frac{\sin(t)}{1-\cos(t)} dt \\ &= \frac{1}{2} \cdot \int_{1+\cos(\pi/2)}^{1+\cos(x^2)} -\frac{1}{u} du + \frac{1}{2} \cdot \int_{1-\cos(\pi/2)}^{1-\cos(x^2)} \frac{1}{u} du \\ &= \frac{1}{2}\left(-\ln\left(|u|\right)\right)\Big|_1^{1+\cos(x^2)} + \frac{1}{2}\left(\ln\left(|u|\right)\right)\Big|_1^{1-\cos(x^2)} \\ &= -\frac{1}{2}\ln\left(\left|1+\cos(x^2)\right|\right) + \frac{1}{2}\ln\left(\left|1-\cos(x^2\right|\right) \\ &= \frac{1}{2}\ln\left(\left|\frac{1-\cos(x^2)}{1+\cos(x^2)}\right|\right).\end{aligned}$$

Combining, we have that

$$\begin{aligned}\text{area} &= \int_{\pi/2}^{x^2} te^t dt - \int_{\pi/2}^{x^2} \frac{1}{\sin(t)} dt \\ &= \left(x^2-1\right)e^{x^2} - \frac{\pi^2-1}{4}e^{\pi^2/4} - \frac{1}{2}\ln\left(\left|\frac{1-\cos(x^2)}{1+\cos(x^2)}\right|\right).\end{aligned}$$

Ugly: yes. But an answer!

(2): So, we *could* just calculate the derivative of the above. But that would be awful! Instead, let's use the first fundamental theorem of calculus (which applies here b/c everything's continuous and integrable and bounded on this domain.)

Specifically, notice that we can write $F(x) = G(x^2)$, where

$$G(x) = \int_{\pi/2}^{x} \left(te^t - \frac{1}{\sin(t)}\right) dt.$$

Then, the chain rule says that

$$F'(x) = 2x \cdot G'(x^2),$$

and the fundamental theorem of calculus says that

$$G'(x) = xe^x - \frac{1}{\sin(x)}.$$

Combining, we have

$$F'(x) = 2x \cdot \left(x^2 e^{x^2} - \frac{1}{\sin(x^2)}\right),$$

which was certainly an easier derivation than calculuating the derivative through brute force! ☐

4. Problem 4: Sequences, Limits, e^x, and L'Hôpital

Question 4.1. *Prove that*

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Proof. First, notice that if we expand $\left(1 + \frac{x}{n}\right)^n$ via the binomial theorem, we have

$$\begin{aligned}\left(1 + \frac{x}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} \\ &= 1 + \binom{n}{1}\frac{x}{n} + \binom{n}{2}\frac{x^2}{n^2} + \ldots + \binom{n}{n}\frac{x^n}{n^n} \\ &= 1 + \frac{n}{1!}\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \ldots + \frac{n!}{n!}\frac{x^n}{n^n} \\ &= 1 + \frac{n}{n}\frac{x}{1!} + \frac{n(n-1)}{n^2}\frac{x^2}{2!} + \ldots + \frac{n!}{n^n}\frac{x^n}{n!}.\end{aligned}$$

From this expansion, we can deduce two things:

(1) Because $\frac{n(n-1)\cdot\ldots\cdot(n-(k-1))}{n^k} \leq \frac{n^k}{n^k} = 1$, we know that this sum is bounded above by the sum $\sum^n \frac{x^k}{k!}$, which is in turn bounded above by the infinte series $\sum^\infty \frac{x^k}{k!}$, which converges by the ratio test.

(2) If we examine the term $\frac{n(n-1)\cdot\ldots\cdot(n-(k-1))}{n^k}$, we can in fact see that these all increase as n increases. Specifically, we can write

$$\begin{aligned}\frac{n(n-1)\cdot\ldots\cdot(n-(k-1))}{n^k} &= \frac{n}{n}\cdot\frac{n-1}{n}\cdot\ldots\cdot\frac{n-(k-1)}{n} \\ &= 1\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right),\end{aligned}$$

and it's clear that increasing n increases the value of this term.

We've just proven that the terms $\left(1 + \frac{x}{n}\right)^n$ form a monotone-increasing sequence that's bounded above. Therefore, it must have a limit! Call this limit y.

We claim that for any x, $\ln(y) = x$ – in other words, that y is an inverse function to ln, and therefore that $y = e^x$ (which is what we want to prove.)

To see this, we examine $\ln(y)$, and use the fact that continuous functions like ln can pass through limits:

$$\begin{aligned}
\ln(y) &= \ln\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right)\\
&= \lim_{n\to\infty}\ln\left(\left(1+\frac{x}{n}\right)^n\right)\\
&= \lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right)\\
&= \lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{1/n}\\
&= \lim_{n\to\infty}\frac{x\ln\left(1+\frac{x}{n}\right)}{x/n}\\
&= x\cdot\lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{x/n}
\end{aligned}$$

So: we now make the substitution $h = x/n$, and switch from evaluating the limit as $n \to \infty$ to looking at the limit as $h \to 0$:

$$\begin{aligned}
\ln(y) &= x\cdot\lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{x/n}\\
&= x\cdot\lim_{n\to\infty}\frac{\ln(1+h)}{h}.
\end{aligned}$$

Because both the top and bottom go to 0 as $h \to 0$, we can use L'Hôpital's rule (or even just the definition of the derivative for ln) to see that

$$\begin{aligned}
\ln(y) &= x\cdot\lim_{n\to\infty}\frac{\ln(1+h)}{h}\\
&= x\cdot\lim_{n\to\infty}\frac{\frac{1}{1+h}}{1}\\
&= x.
\end{aligned}$$

So $\ln(y) = x$, for any x : i.e. $y = e^x$, as claimed. $\square$