MATH 8, SECTION 1, WEEK 8 - RECITATION NOTES

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Abstract. These are the notes from Friday, Dec. 3rd's lecture. In this talk, we discuss partial fractions and complex power series.

1. Random Question

Question 1.1. *Oh noes! You've just knocked over your drink and stained the tablecloth. Through careful measurements, you realize that the stain has diameter 1. Can you cover it with your salad plate (which has radius $1/\sqrt{3}$?)*

(Bonus question: suppose you're quite clumsy and knock over another drink during dessert, when the only plates on hand have radius $< 1/\sqrt{3}$. Are you doomed?)

2. Administrivia

- Final-review-thing: will happen Monday, from 2-3pm!
- Math 8 notes and the final: Just like with the midterm, the online Math 8 notes are fair game to be consulted for the final, but cannot be cited as a direct source. So you're allowed to use them, but you can't say "By the online notes for Math 8 on 11/29/10, we know that **(thing)** holds."

So: today's lecture is in two parts, as it's covering material from Wednesday (when I had the plague and had to cancel class) and today's lectures. The two topics we'll discuss here are (1) the use of partial fractions in calculus, and (2) complex power series and their radii of convergence.

3. Partial Fractions

The method of partial fractions is an algebraic trick designed to find the integrals of things like

$$\int \frac{p(x)}{q(x)} dx,$$

where $p(x)$ and $q(x)$ are polynomials, $q(x) \neq 0$. Specifically: to calculate the indefinite integral above, the method proceeds as follows:

(1) By using polynomial long division, transform $\frac{p(x)}{q(x)}$ into $p_1(x) + \frac{p_2(x)}{q(x)}$, where the degree of $p_2(x)$ is strictly smaller than that of $q(x)$.

(2) Factor $q(x)$ into irreducible polynomials

$$q(x) = \left((r_1(x))^{s_1} \cdot \ldots \cdot (r_m(x))^{s_m}\right) \cdot \left((t_1(x))^{u_1} \cdot \ldots \cdot (t_n(x))^{u_n}\right),$$

where the $r_i(x)$'s are irreducible polynomials of degree 1 (i.e. your $x - 2$ terms) and the $t_k(x)$'s are irreducible polynomials of degree 2 (i.e. your $x^2 + x + 1$ terms.)

(3) Find constants $A_{i,j}, B_{k,l}, C_{k,l}$ that solve the given equation:

$$\frac{p_2(x)}{q(x)} = \sum_{i=1}^{m}\sum_{j=1}^{s_i} \frac{A_{i,j}}{(r_i(x))^j} + \sum_{k=1}^{n}\sum_{l=1}^{u_k} \frac{B_{k,l}x + C_{k,l}}{(t_k(x))^l}.$$

Equivalently, if you multiply through by $q(x)$, you're trying to find constants that solve the equation

$$p_2(x) = \sum_{i=1}^{m}\sum_{j=1}^{s_i} A_{i,j}\frac{q(x)}{(r_i(x))^j} + \sum_{k=1}^{n}\sum_{l=1}^{u_k} (B_{k,l}x + C_{k,l})\frac{q(x)}{(t_k(x))^l},$$

where all of the fractions on the inside become polynomials after dividing through (because the $r_i(x)$'s and $t_k(x)$'s are factors of $q(x)$!)

(4) To do this last step, simply group the terms on the right by their factors of x, so that you have (say) $A_{1,1} + B_{2,3}$ copies of x^2 and $4C_{2,1}$ copies of x. Then, setting this equal to the left-hand side gives you $\deg(p_2(x))$-many linear equations to solve – one for every power of x, up to $\deg(p_2(x))$.

(5) Solving these equations tells us that

$$\int \frac{p(x)}{q(x)}dx = \int \left(p_1(x) + \sum_{i=1}^{m}\sum_{j=1}^{s_i} \frac{A_{i,j}}{(r_i(x))^j} + \sum_{k=1}^{n}\sum_{l=1}^{u_k} \frac{B_{k,l}x + C_{k,l}}{(t_k(x))^l}\right) dx$$

By using u-substitutions, the power rule, and trig substitutions, the above is (usually) easy to integrate. Do so and you're done!

Like many subjects in calculus, this is a topic that's probably made a lot clearer through examples:

Question 3.1. *Calculate*

$$\int \frac{x^4+1}{x(x^2+1)^2}dx.$$

Proof. We proceed by the method of partial fractions. Because the degree of the denominator is greater than that of the numerator, we don't need to perform polynomial long division to simplify the numerator. Similarly, the denominator is already factored: so we can skip to step 3, where we're trying to solve for variables A, B, C, D, E such that

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2};$$

i.e., if we multiply through by $x(x^2+1)^2$,

$$x^4+1 = A(x^2+1)^2 + (Bx+C)\cdot(x(x^2+1)) + (Dx+E)\cdot x.$$

If we expand the right-hand side and group together terms by their powers of x, we have that the above equation is in fact

$$\begin{aligned} x^4+1 &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A. \end{aligned}$$

The above is equivalent to the following five equations:

$$
\begin{aligned}
1x^4 &= Ax^4 + Bx^4 \\
0x^3 &= Cx^3 \\
0x^2 &= 2Ax^2 + Bx^2 + Dx^2 \\
0x &= Cx + Ex \\
1 &= A
\end{aligned}
$$

Solving these equations gives us that $A = 1, B = 0, C = 0, D = -2, E = 0$ and thus that

$$\int \frac{x^4+1}{x(x^2+1)^2} dx = \int \frac{1}{x} - \frac{2x}{(x^2+1)^2} dx.$$

Finally, if we use the u-substitution $u = x^2 + 1$ in the second integral, we get that

$$\int \frac{x^4+1}{x(x^2+1)^2} d = \ln |x| + \frac{1}{x^2+1} + C.$$

So we're done! □

To further illustrate this method, we study another example:

Question 3.2. *Calculate*

$$\int \frac{x^4 + x^3 + 4x^2}{x^3 - 1} dx.$$

Proof. We proceed via partial fractions. First, we use polynomial long division to divide the numerator by the denominator:

$$
\begin{array}{rl}
 & x+1 \\
x^3 - 1 \big) & \overline{x^4 + x^3 + 4x^2 } \\
 & \underline{-x^4 + x} \\
 & \phantom{-x^4+{}} x^3 + 4x^2 + x \\
 & \underline{-x^3 + 1} \\
 & \phantom{-x^4+x^3+{}} 4x^2 + x + 1
\end{array}
$$

Consequently, we have that

$$\int \frac{x^4 + x^3 + 4x^2}{x^3 - 1} dx = \int \left(x + 1 + \frac{4x^2 + x + 1}{x^3 - 1} \right) dx,$$

and that it suffices to turn our attention to the fraction $\frac{4x^2+x+1}{x^3-1}$.

If we proceed by the method of partial fractions, we first factor $x^3 - 1$ as $(x-1)(x^2+x+1)$, and seek to find constants A, B, C such that

$$\frac{4x^2 + x + 1}{x^3 - 1} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1};$$

i.e. A, B, C such that

$$\begin{aligned} 4x^2 + x + 1 &= A(x^2 + x + 1) + (Bx + C)(x - 1) \\ &= (A + B)x^2 + (A - B + C)x + (A - C). \end{aligned}$$

Setting $A = 2, B = 2, C = 1$ solves the above equation, and allows us to write

$$\int \frac{x^4 + x^3 + 4x^2}{x^3 - 1} dx = \int \left(x + 1 + \frac{2}{x - 1} + \frac{2x + 1}{x^2 + x + 1} \right) dx,$$

This can be integrated by splitting the integral up into parts, and using the two (distinct) u-substitutions $u = x - 1$ and $u = x^2 + x + 1$ on the two fractions. Doing so gives us our answer:

$$\begin{aligned} \int \frac{x^4 + x^3 + 4x^2}{x^3 - 1} dx &= \frac{x^2}{2} + x + 2\ln(|x - 1|) + \ln(|x^2 + x + 1|) + C \\ &= \frac{x^2}{2} + x + \ln(|x - 1|^2) + \ln(|x^2 + x + 1|) + C \\ &= \frac{x^2}{2} + x + \ln\left(|x - 1|^2 \cdot |x^2 + x + 1|\right) + C \\ &= \frac{x^2}{2} + x + \ln\left(|x^4 - x^3 - x + 1|\right) + C. \end{aligned}$$

□

4. Complex Power Series

On Wednesday (while I was sick,) the Math 1 lecture discussed the concepts of series and sequences of *complex* numbers, and how convergence works in these situations. Today, I'd like to discuss a particular special case of series of complex numbers: that of **complex power series**, i.e. things of the form

$$\sum_{n=0}^{\infty} \alpha_n z^n,$$

where the α_n's are all complex numbers.

Specifically, the question we're interested in is is the following: given some complex power series $\sum_{n=0}^{\infty} \alpha_n z^n$, for what values of $z \in \mathbb{C}$ does this series converge? For real numbers, we had the remarkably useful concept of a **radius of convergence**: for any given power series $\sum_{n=0}^{\infty} a_n x^n$ of real numbers, we knew that there was some value R such that this series would converge for any $|x| < R$ and diverge for any $|x| > R$.

Could such a thing exist for the complex numbers? As it turns out: yes! We in fact have the following theorem:

Theorem 4.1. *For any complex power series*

$$\sum_{n=0}^{\infty} \alpha_n z^n,$$

there is some constant $R \in [0, \infty]$ such that this power series converges for any $z \in \mathbb{C}$ with $|z| < R$, and diverges at any value of z with $|z| > R$.

Proof. (We omitted this proof from lecture due to time constraints; it's reproduced here in all of its glory.)

Take any complex-valued power series

$$\sum_{n=0}^{\infty} \alpha_n z^n,$$

and suppose that it converges at some point $z_0 \neq 0 \in \mathbb{C}$. (If no such point exists, then its radius of convergence is simply 0.)

We then claim that for any $z \in \mathbb{C}$ with $|z| < |z_0|$, our complex power series converges at z. To see this: notice that because the series

$$\sum_{n=0}^{\infty} \alpha_n {z_0}^n$$

converges, the individual terms of this sum (the $\alpha_n x_0^n$'s) must converge to 0. Consequently, we know that these terms must be bounded: i.e. that there is some value of M such that

$$|\alpha_n {z_0}^n| \leq M,$$

for any n.

Given this observation, we then have that for any z with $|z| < |z_0|$,

$$\begin{aligned} |\alpha_n z^n| &= \left| \alpha_n {z_0}^n \cdot \left(\frac{z}{z_0} \right)^n \right| \\ &= |\alpha_n {z_0}^n| \cdot \left| \frac{z}{z_0} \right|^n \\ &\leq M \cdot \left| \frac{z}{z_0} \right|^n. \end{aligned}$$

But this quantity $|\alpha_n z^n|$ is greater than both the real and imaginary parts of $\alpha_n z^n$! Consequently, if we use the first comparison test and show that the series

$$\sum_{n=0}^{\infty} M \cdot \left| \frac{z}{z_0} \right|^n$$

converges, we're done! (as this means that both the real and imaginary parts of our series converge.) But this is trivial: because $|z| < |z_0|$, we know that $\left| \frac{z}{z_0} \right|$ is less than 1: therefore, the above series is simply a geometric series and must converge.

We've thus proven that if our power series converges at any value z_0, it must converge at any value z with $|z| < |z_0|$.

Let T be the collection of points in $\mathbb{C}$ for which our power series converges. Examine the absolute value of all of the points in T. There are two possibilities:

- The absolute value of elements in T is unbounded: i.e. there are arbitrarily large complex numbers for which our power series converges. Then, by our earlier work, we know that this power series must converge on all of $\mathbb{C}$, as if our power series converges at any z_0, it must converge at any value z with $|z| < |z_0|$. In other words, our power series has ∞ as its radius of convergence.

- There is a supremal value R for the set $\{|z| : z \in T\}$. In this case, we know (by the result proven earlier) that our power series must converge for any z with $|z| < R$. Furthermore, we know that for any z with $|z| > R$, z cannot be in the set T, because R was a supremum: so for any z with $|z| > R$, we must have that our power series diverges. In other words: our power series has R as its radius of convergence.

But this is exactly what we wanted to prove! So we're done: we've shown that complex power series have radii of convergence, just as we claimed. $\square$

The cool thing about this theorem is it tells us the following: to understand the radius of convergence of a complex power series, if all of its coefficients are real, it suffices to simply know the radius of convergence of the corresponding real power series! In other words, suppose we have a complex power series

$$\sum_{n=0}^{\infty} \alpha_n z^n,$$

where all of the α_n's are real numbers. Suppose furthermore that the radius of convergence of the **real-valued** power series with the same coefficients,

$$\sum_{n=0}^{\infty} \alpha_n x^n,$$

was R. Then, look at the complex power series again; by our theorem above, we know that not only must it have a radius of convergence, it must also be R – the same as in the real case! This is because if our complex power series converges at some real point $x > 0$, we've proven that it has to converge at every single complex point z with $|z| < x$; consequently, the radius of convergence in the real case must be the same as that for the complex case!

We state this in a theorem for added emphasis:

Theorem 4.2. *If*

$$\sum_{n=0}^{\infty} \alpha_n z^n$$

is a complex power series where all of the α_n's are real-valued, then the radius of convergence of $\sum_{n=0}^{\infty} \alpha_n z^n$ is the same as that of the real power series

$$\sum_{n=0}^{\infty} \alpha_n x^n.$$

This is remarkably useful, as we can use tools like the ratio test and comparison test to work with real power series (in the complex case, we don't have a lot of those theorems!) To illustrate this, we work one quick example:

Question 4.3. *What is the radius of convergence of the complex power series*

$$\sum_{n=0}^{\infty} 5^n z^{n!}?$$

Proof. By our above discussion, it suffices to simply find the radius of convergence of the real power series

$$\sum_{n=0}^{\infty} 5^n x^{n!}.$$

To do this, pick any positive value of x and apply the ratio test:

$$\lim_{n\to\infty} \frac{5^{n+1}x^{(n+1)!}}{5^n x^{n!}} = \lim_{n\to\infty} 5 \cdot x^{(n+1)!-n!} = \lim_{n\to\infty} 5 \cdot x^{n\cdot n!} = \begin{cases} 0, & x < 1, \\ 5, & x = 1, \\ \infty, & x > 1. \end{cases}$$

Consequently, this real-valued power series has 1 as its radius of convergence; by our earlier discussion, this must be the radius of convergence of the complex power series as well. □

It bears noting that – just as in the real case! – knowing a complex power series's radius of convergence doesn't tell you anything about what happens for values of z with magnitude **equal** to that radius. For example, the three power series

$$\sum_{n=0}^{\infty} z^n, \qquad \sum_{n=0}^{\infty} \frac{z^n}{n^2}, \qquad \sum_{n=0}^{\infty} \frac{z^n}{n}$$

all have 1 as their radius of convergence. Yet, the first power series diverges at any z with $|z| = 1$, the second converges on any z with $|z| = 1$, and the third converges for some values with absolute value 1 and diverges at others (specifically, it diverges at $z = 1$ and converges everywhere else.)