MATH 8, SECTION 1, WEEK 9 - RECITATION NOTES

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Abstract. These are the notes from Monday, Nov. 22nd's lecture, where we started our discussion of Taylor series.

1. Random Question

Question 1.1. *A sequence of symbols is* ***repetition-free*** *if it never contains the same segment twice: i.e. it never has any "22"'s, nor any "2121"'s, nor any "213213"'s in it anywhere.*

Is there an infinite, repetition-free sequence that uses only the three symbols (1,2,3)?

2. Polar Coördinates and the Complex Plane

In class today, we introduced the complex plane $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$, and asked the following question: how can we define functions on $\mathbb{C}$?

Specifically: in $\mathbb{R}$, the functions $e^x, \sin(x), \cos(x)$ were remarkably useful to have around. Is there any way to extend the definitions of these functions to the complex plane?

At a first glance, it may not seem like there is a good way to do this: after all, we defined $\sin(x)$ in a strictly geometric fashion using triangles, and defined e^x as the inverse of $\ln(x)$, which was the integral of $1/t$ from 1 to x. Extending these kinds of definitions to $\mathbb{C}$ seems impossible: what would we mean by taking an integral from 1 to $2 + 3i$, or a triangle with side length i?

So: if we want to define these functions for $\mathbb{C}$, we need to come up with another way of defining them. How can we do that?

The answer, as it turns out, is Taylor series! Specifically: last week, we showed that

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots, \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots, \text{and} \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots
\end{aligned}$$

for all real x. Consequently, we can choose to **define**

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots,$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots, \text{and}$$
$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots,$$

for all $z \in \mathbb{C}$.

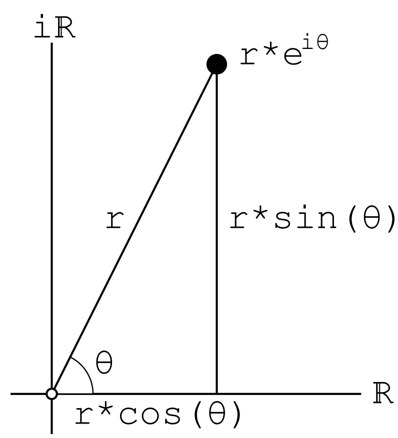
This definition has some remarkably beautiful consequences. For example, if we plug in iz into the power series for e^z, we have that

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \ldots \\ &= 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} \ldots \\ &= \left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots\right) \\ &= \cos(z) + i\sin(z); \end{aligned}$$

in other words, that $e^{iz} = \cos(z) + i\sin(z)$. If we let $z = \pi$, this gives us **Euler's formula**:

$$e^{i\pi} + 1 = 0.$$

More generally, these definitions give us an incredibly beautiful way to visualize the complex plane with polar coördinates! Specifically, examine the complex point $r \cdot e^{i\theta}$. By our formula above, we can write this point as $r\cos(\theta) + i \cdot r\sin(\theta)$, which is the following point in the complex plane:



In other words: if a point in $\mathbb{C}$ has polar coördinates (r, θ), then it *is* the point $re^{i\theta}$! The upshot of this is that points in the complex plane have remarkably simple polar coördinates: often, when working in $\mathbb{C}$, it can be a lot easier to manipulate points by thinking of them as of the form $re^{i\theta}$ rather than of the form $a + bi$.

The following section on "roots of unity" is an excellent example of such a situation, where an otherwise intractable problem on $\mathbb{C}$ is made trivial through the use of polar coördinates:

3. Roots of Unity

Over the real numbers, the equation

$$x^n - 1 = 0$$

had only the root 1, if n was odd, and $\{1, -1\}$ if n was even.

In the complex plane, however, the situation is much more complicated; in specific, by the fundamental theorem of algebra, we know that the equation

$$z^n - 1 = 0$$

must have n solutions.

What are they? Well: if we express z in polar coördinates as $re^{i\theta}$, we can see two quick things:

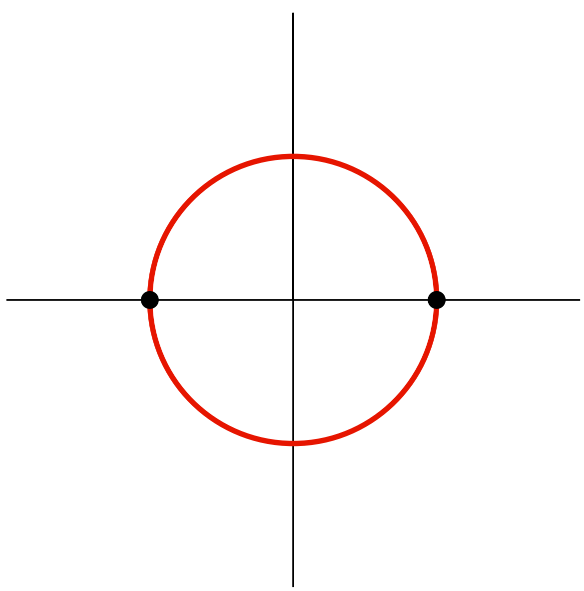
- $r = 1$. This is because $|r^n \cdot e^{in\theta}|$ is just r^n, and the only positive number r such that $r^n = 1$ is 1.
- $\theta = k\frac{2\pi}{n}$, for some k. To see this: simply use Euler's formula to write $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$. If this expression is equal to 1, we need to have $\cos(n\theta) = 1$ and $i\sin(n\theta) = 0$ (so that the imaginary and real parts line up!) – in other words, we need $n\theta$ to be a multiple of 2π.

Combining these two results then tells us that the n distinct roots of $z^n - 1 = 0$ are

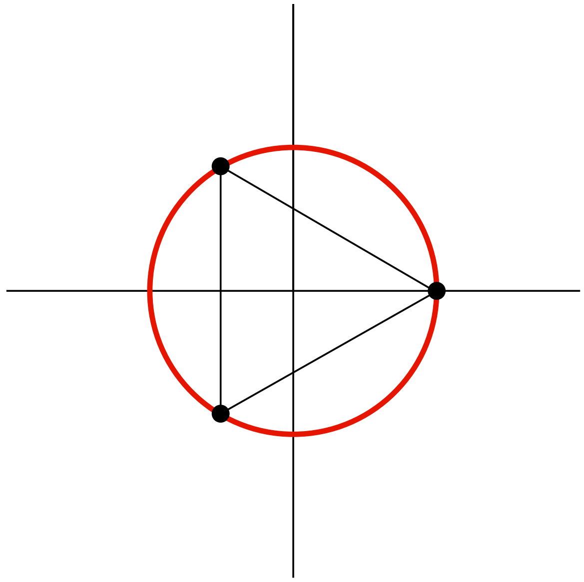
$$e^0, e^{\frac{2\pi}{n}}, e^{2\frac{2\pi}{n}}, e^{3\frac{2\pi}{n}} \dots, e^{(n-1)\frac{2\pi}{n}}.$$

So: in some explicit cases, what are these roots?

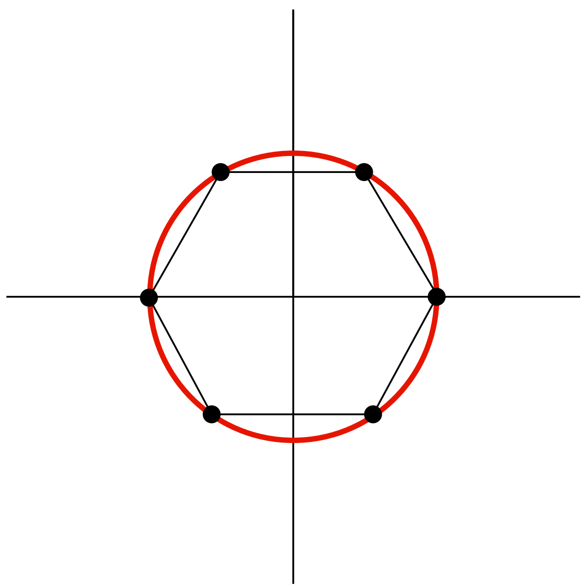
Example 3.1. The second roots of unity are, by the above, $e^0 = 1$ and $e^{\frac{2\pi}{2}} = e^{\pi} = \cos(\pi) + i\sin(\pi) = -1$, and can be graphed on the unit circle $|z| = 1$ as shown below:



Example 3.2. The third roots of unity are simply (by the above) the points $e^0, e^{\frac{2\pi}{3}},$, and $e^{\frac{4\pi}{3}}$; their graph is the three-equally-spaced points on the unit circle shown below.



Example 3.3. The sixth roots of unity are the points $e^0, e^{\frac{2\pi}{6}}, e^{\frac{4\pi}{6}}, e^{\frac{6\pi}{6}}, e^{\frac{8\pi}{6}}$ and $e^{\frac{10\pi}{6}}$, and form the hexagon inscribed in the unit circle displayed below:



In the above pictures, these n-th roots of unity always correspond to the vertices of a regular n-gon inscribed in the unit circle. As it turns out, this is always true: a quick proof of this statement is just noticing that

(1) we get all of our n-th roots of unity by starting at 1 and rotating by $\frac{2\pi}{n}$ around the unit circle, and

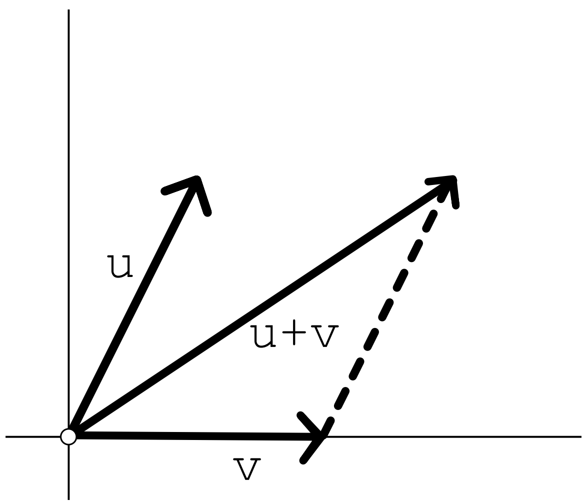
(2) doing this process creates n evenly-spaced points on the unit circle – i.e. the vertices of a n-gon.

The basic idea used above has a quick and remarkable consequence:

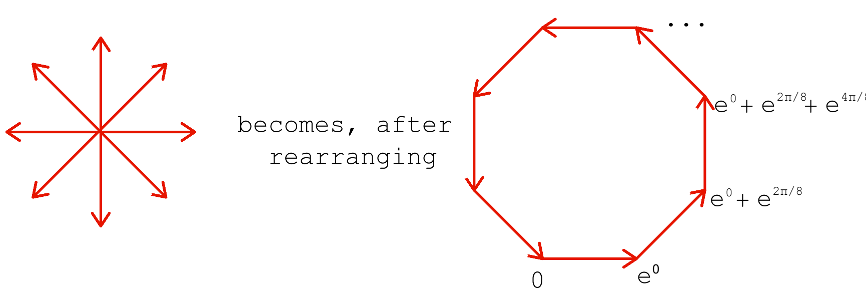
Theorem 3.4. *The sum of all of the n-th roots of unity is 0, for any $n \geq 2$.*

Proof. We start by stating something that –algebraically – is painfully trivial, but visually is much less so:

Proposition 3.5. *The sum of any two points (a, b) and (c, d) in the plane is just $(a + c, b + d)$. In other words: if u and v are a pair of vectors based at the origin, then we can get the vector $u + v$ by placing the start of u at the tip of v, as shown below:*



Given this idea, we can visualize adding up the roots of unity in the following way: simply start with the vector made by the point e^0 and the origin, and add in sequence the vectors formed by the points $e^{k\frac{2\pi}{n}}$. As we discussed above, and is visually apparent in the picture below, these are all vectors of length 1 at angles $2\pi k/n$; so, adding them up visually creates a n-gon. But what does this mean about their sum? Well, that if we add all of these vectors together, we return to where we started. But the only number that has this property is 0 – so their sum is 0, as claimed.



$\square$

The above is a rather unexpected property, and raises perhaps a parallel question: if their sum has such an odd property, what happens to their product? We answer this question with the following theorem:

Theorem 3.6. *The product of all of the n-th roots of unity is $(-1)^{n+1}$, for any n.*

Proof. So: first, begin by writing all of the n-th roots of unity in the form $\left(e^{\frac{2\pi}{n}}\right)^k$, where k can range from 1 to n. Then, we have that the product of all of the n-th roots of unity is just

$$\begin{aligned}
\prod_{k=0}^{n-1}\left(e^{\frac{2\pi}{n}}\right)^k &= \exp\left(\frac{2\pi}{n}\cdot\sum_{k=1}^{n} k\right)\\
&= \exp\left(\frac{2\pi}{n}\cdot\frac{n(n+1)}{2}\right)\\
&= e^{(n+1)\pi}\\
&= (-1)^{n+1},
\end{aligned}$$

where the above steps were done by using the rules of multiplication and exponentiation, and Euler's summation formula. (for those of you who've forgotten: $\exp(x)$ is just the function e^x, and is used whenever actually writing a bunch of things in the exponent would render the mathematics unreadable.) $\square$