MATH 8, SECTION 1, WEEK 1 - RECITATION NOTES

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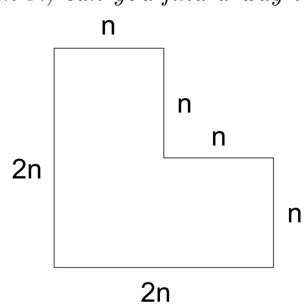
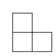
ABSTRACT. These are the notes from Wednesday, Sept. 29th's lecture. In this talk, we continue our discussion on proofs by contradiction, and examine proofs by induction.

1. Random Question

Question 1.1. *For any n in $\mathbb{N}$, can you find a way to tile*[1] *the shape*



with triominoes made out of three 1x1 squares of the form  *?*

2. Administrivia and Announcements

- Class times will remain at 2-3 MWF for the quarter; while some other time slots are marginally better, none of them are sufficiently better to clear a class change with the Registrar (and running two sections appears to be an impossibility as well.) Sorry, CS1 students! As always, any students who can't make Math 8 consistently are welcome to attend as unregistered students when they can, and read the course notes online.
- Homework policy! In your Math 1 HW,
  - you **cannot** cite the Math 8 notes or lectures! Your section TA will have no idea what you're talking about.
  - you **can** use your Math 8 notes for reference and inspiration! I.e. direct copying of the notes = bad, but using them to remember how a proof goes or some clever trick, and then writing up the results in your own words = good.

[1] A tiling of some region R with some shape S is a way of covering all of the points in R with translations, rotations, and reflections of the shape S, so that (1) all of the copies of S lie inside of R, and (2) none of the copies of S overlap (except for possibly on their boundaries.)

3. Proofs by Contradiction, Part II

Last time, we described proofs by contradiction as the following process: suppose we want to prove that some statement P is true. How can we do this? Well, there are only two possiblities: either P is true, or$\neg P$ is. So: if we show that $\neg P$ is impossible – in other words, that assuming $\neg P$ leads to a contradiction – then by process of elimination we have tht P must hold!

On Monday, we gave one example of a proof by contradiction; today, we have two more proofs to further illustrate the method.

Lemma 3.1. *(Euclid) There are infinitely many prime numbers.*

Proof. (N.b.: for the entertainment of those who've already encountered Euclid's proof about prime numbers, the proof below is written in the "way of prime" – i.e. all of the words below contain prime numbers of letters. Silly, yes. But fun!)

Suppose not: in other words, suppose n prime numbers exist, for $n \in \mathbb{N}$. Label these prime numbers

$$p_1, p_2, \ldots p_n,$$

and examine the product

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n.$$

For any p_i, the product $p_1 \cdot \ldots \cdot p_n$ divided by p_i is an integer. So: if we add one to the product

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n,$$

we get a sum

$$(p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$$

for which all of the p_i's are *not* factors! So, there are two possibilities:

(1) $(p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$ is prime (and not in our set – contradiction!), or
(2) $(p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$ has to contain new prime factors (again, these aren't in our set – contradiction!)

So there are infinitudes of prime numbers! $\square$

. . . yeah, I'm a nerd. Here's another proof by contradiction!

Lemma 3.2. *If a, b, and c are all odd integers, then there cannot be a rational solution to the equation*

$$ax^2 + bx + c = 0.$$

Proof. Suppose not: that there is some rational number $p/q \in \mathbb{Q}$ with $\mathrm{GCD}(p, q) = 1$ and

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0.$$

Then, if we were to multiply through by q^2, we would have

$$ap^2 + bpq + cq^2 = 0.$$

Then, because $\mathrm{GCD}(p, q) = 1$, there are only three possibilities:

(1) p is even and q is odd. In this case, the left hand side of our above equation is (if we only consider the parity[2])

$$(\text{odd} \cdot \text{even} \cdot \text{even}) + (\text{odd} \cdot \text{even} \cdot \text{odd}) + (\text{odd} \cdot \text{odd} \cdot \text{odd}) = \text{even} + \text{even} + \text{odd} = \text{odd},$$

and thus not equal to zero.

(2) p is odd and q is even. In this case, similar parity arguments to the above force the quantity $ap^2 + bpq + cq^2$ to again be odd, and thus not equal to zero.

(3) Both p and q are odd. In this final case, we again have that the quantity $ap^2 + bpq + cq^2$ is odd, and thus nonzero.

In all three cases, we arrived at a contradiction; thus, we know that no such rational number can exist! □

These examples, one hopes, have illustrated some of the power and utility in proofs by contradiction; however, it bears noting that contradiction is not *always* the way to prove something! Consider the following cautionary proof:

Lemma 3.3. $0 \cdot x = 0$, *for any real number* x.

Proof. Suppose (for contradiction) that there is some x such that $0 \cdot x \neq 0$. However, as

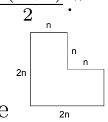
$$\begin{aligned}
0 \cdot x &= (0+0) \cdot x && \text{(axiom of additive identity)} \\
&= 0 \cdot x + 0 \cdot x && \text{(distributive axiom)} \\
\Rightarrow 0 \cdot x - 0 \cdot x &= 0 \cdot x + 0 \cdot x - 0 \cdot x && \text{(existence of additive inverses)} \\
\Rightarrow 0 &= 0 \cdot x + 0 && \text{(definition of additive inverses)} \\
&= 0 \cdot x && \text{(definition of additive identity)}
\end{aligned}$$

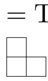
So $0 = 0 \cdot x$, which contradicts our assumption that $0 \neq 0 \cdot x$. So we've proven that $0 \cdot x = 0$. □

The flaw in the above proof was that **contradiction was compeletely unnecessary!** In fact, if we omit the first and second to last sentences, we would have a normal, direct proof of our claim; the only thing that our "proof by contradiction" language gave us was extra words and confusing phrasing. So, when you can prove something directly, do so! Proofs by contradiction are usually best attempted in situations where it seems difficult to construct a solution, or when you're dealing with negative claims (i.e. that no object exists to satisfy some given property.)

4. Proofs by Induction

Sometimes, in mathematics, we will want to prove the truth of some statement $P(n)$ that depends on some variable n. For example:

- $P(n) =$ "The sum of the first n natural numbers is $\frac{n(n+1)}{2}$."
- $P(n) =$ This week's random question: i.e. "The shape 

 can be tiled with -triominoes."

[2]the **parity** of a number is whether it is odd or even.

- $P(n) =$ "The number of ways to divide n distinct objects into k nonempty piles is

$$\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

For a given n, we can use our previously-established methods to prove the truth or falsity of the statement: but what if we wanted to prove such a statement for **every** natural number? How would we do that?

Well: let's explore an example, to get the flavor of how we might try doing this.

Lemma 4.1.

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1},$$

for any real number $x \neq 1$ *and* $n \in \mathbb{N}$.

Proof. So: for $n = 1$, this is pretty straightforward, right? In this case, we specifically have that our sum is

$$\sum_{k=0}^{1-1} x^k = x^0 = 1 = \frac{x-1}{x-1}.$$

Nothing fancy; just some algebraic manipulations.

What about for $n = 2$? Well, in this case, we have that our sum is

$$\sum_{k=0}^{2-1} x^k = x^0 + x^1.$$

From our earlier work with $n = 1$, we know that $x^0 = \frac{x-1}{x-1}$. So, if we plug that in and do some basic algebra, we have that

$$\begin{aligned}
x^0 + x^1 &= \left(\frac{x-1}{x-1}\right) + x \\
&= \left(\frac{x-1}{x-1}\right) + x \cdot \left(\frac{x-1}{x-1}\right) \\
&= \frac{x^2 + x - x + 1}{x-1} \\
&= \frac{x^2+1}{x-1},
\end{aligned}$$

and thus that our claim holds for $n = 2$ as well.

What about $n = 3$? Well, by this point we're beginning to see the pattern: if we use our results from the $n = 2$ case and the same algebraic tricks, we can see that

$$\begin{aligned}\sum_{k=0}^{3-1} x^k &= x^0 + x^1 + x^2 \\ &= \left(\frac{x^2-1}{x-1}\right) + x^2 \\ &= \left(\frac{x^2-1}{x-1}\right) + x^2 \cdot \left(\frac{x-1}{x-1}\right) \\ &= \frac{x^3 + x^2 - x^2 + 1}{x-1} \\ &= \frac{x^3+1}{x-1}.\end{aligned}$$

So, for $n = 4$, we can just repeat, right? And so on and so on, for every n? $\square$

Proofs by induction are, at their heart, a way of formalizing the "just repeat" statement above. Their formal structure goes as follows:

Lemma 4.2. *$P(n)$ holds, for all $n \geq k$.*

Proof. We proceed by induction.

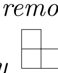
Base case: we prove (by hand) that $p(k)$ holds. In some situations, we may also prove that $p(k+1), p(k+2)$, and a few more cases hold, if it turns out to be necessary; but typically we will only need to prove the one case.

Inductive step: Assuming that $p(m)$ holds for all $k \leq m < n$, we prove that $p(n)$ holds.

Conclusion: $P(n)$ holds for all $n \geq k$. $\square$

The best analogy for what we're doing here is perhaps to dominoes: consider each of your $P(n)$ propositions as individual dominoes – one labeled $P(1)$, one labeled $P(2)$, one labeled $P(3)$, and so on/so forth. With our inductive step, we are insuring that all of our dominoes are *lined up* – in other words, that if one of them is true, that it will "knock over" all of the ones after it into being true as well! Then, we can think of the base step as pushing over the first domino; once we do that, the inductive step makes it so that the truth of our initial proposition forces all of the ones after it into being true as well!

If domino analogies aren't your thing, consider instead the following proof:

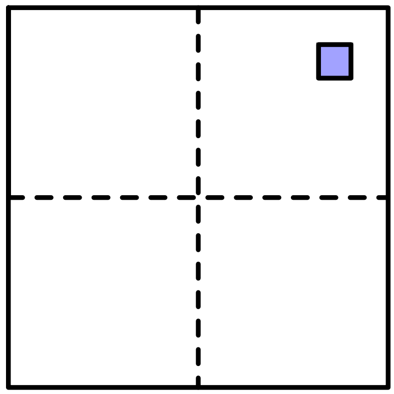
Lemma 4.3. *For any $n \in \mathbb{N}$, take a $2^n \times 2^n$ grid of unit squares, and remove one square from somewhere in your grid. The resulting shape can be tiled by*  *-triominoes.*

Proof. As suggested by the section title, we proceed by induction.

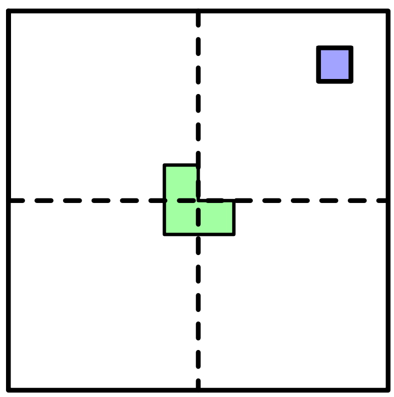
Base case: for $n = 1$, we simply have a 2×2 grid with one square punched out. As this *is* one of our triominoes, we are trivially done here.

Inductive step: Assume that we can do this for a $2^k \times 2^k$-grid without a square, for any $k \leq n$. We then want to prove that we can do this for a $2^{n+1} \times 2^{n+1}$ grid minus a square.

So: take any such grid, and divide it along the dashed indicated lines into four $2^n \times 2^n$ grids. By rotating our grid, make it so that the one missing square is in the upper-right hand corner, as shown below:



Take this grid, and carefully place down one triomino as depicted in the picture below:



Now, look at each of the four $2^n \times 2^n$ squares in the above picture. They all are missing exactly one square: the upper-right hand one because of our original setup, and the other three because of our placed triomino. Thus, by our inductive hypothesis, we know that all of these squares can also be tiled! Doing so then gives us a tiling of the whole shape; so we've created a tiling of the $2^{n+1} \times 2^{n+1}$ grid!

As this completes our inductive step, we are thus done with our proof by induction. $\square$

The above question – one where you are in some sense "growing" or "extending" a result on smaller values of n to get to larger values of n – is precisely the kind of question that induction is set up to solve. The following problem about the Fibonacci numbers, while more in the vein of number theory, is another such "extending" problem:

Definition 4.4. The n-th **Fibonacci number** is defined recursively as follows:

- $f_0 = 0$.
- $f_1 = 1$.
- $f_n = f_{n-2} + f_{n-1}$.

Example 4.5. Here is the start of the Fibonacci number sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Lemma 4.6. *The n-th Fibonacci number is even iff n is a multiple of 3.*

Proof. We proceed by induction.

Base case: This is trivially true for $f_0 = 0, f_1 = 1$, and $f_2 = 1$.

Inductive step: Suppose that for every $0 \leq k < 3n$, that the n-th Fibonacci number is even iff it is a multiple of 3. We seek to show that this is true for all $0 \leq k < 3(n+1)$.

But proving this is just a simple parity calculation: by our inductive hypothesis, we know that

$$\begin{aligned}
f_{3n} &= f_{3n-1} + f_{3n-2} = \text{odd} + \text{odd} = \text{even}, \\
f_{3n+1} &= f_{3n} + f_{3n-1} = \text{even} + \text{odd} = \text{odd}, \\
f_{3n+2} &= f_{3n+1} + f_{3n} = \text{odd} + \text{even} = \text{odd},
\end{aligned}$$

and thus that our property holds for all values $\leq 3(k+1)$.

Thus, by induction, we know that f_n is even iff it is a multiple of 3. $\square$