MATH 8, SECTION 1, WEEK 2 - RECITATION NOTES

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Abstract. These are the notes from Friday, Oct. 8th's lecture. In this talk, we study sequences.

1. Random Question

Question 1.1. *First, prove that you cannot cover $\mathbb{R}$ with disjoint circles of positive radii. Then, find a way to cover $\mathbb{R}^3$ with disjoint circles of positive radii!*

2. Sequences: Working from the Basics

In our last lecture, we introduced the notion of convergence:

Definition 2.1. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value λ if, for any distance ϵ, the a_n's are eventually within ϵ of λ. To put it more formally, $\lim_{n\to\infty} a_n = \lambda$ iff for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n must be within ϵ of our limit λ.

In symbols:

$$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

Most people are generally pretty good with developing an "intuition" for what convergence means; when it comes to actually proving that a sequence converges, however, it's easy to get confused. How do you find your N? What does it mean to have actually proved convergence?

In general, proofs that a given sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value L will go as follows:

- First, examine the quantity $|a_n - L|$, and try to come up with a very simple upper bound that depends on n and goes to zero. Example bounds we'd love to run into: $1/n, 1/n^2, 1/\log(\log(n))$.
- Using this upper bound, given $\epsilon > 0$, determine a value of N such that whenever $n > N$, our simple bound is less than ϵ.
- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $n > N$, $|a_n - L| < \epsilon$.

We work one example of this method here:

Claim 2.2.

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0.$$

Proof. As suggested above, let's examine the quantity $|\sqrt{n+1}-\sqrt{n}-0|$.

$$\begin{aligned}|\sqrt{n+1}-\sqrt{n}-0| &= \sqrt{n+1}-\sqrt{n}\\ &= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}-\sqrt{n}}\\ &= \frac{n+1-n}{\sqrt{n+1}-\sqrt{n}}\\ &= \frac{1}{\sqrt{n+1}-\sqrt{n}}\\ &< \frac{1}{\sqrt{n}}.\end{aligned}$$

This looks rather simple: so let's see if we can use it to find a value of N.

Take any $\epsilon < 0$. If we want to make $\frac{1}{\sqrt{n}} < \epsilon$, we merely need to pick N such that $\frac{1}{\sqrt{N}} < \epsilon$, and then select $n > N$.

This then tells us that for any $\epsilon > 0$, we can find a N such that for any $n > N$, we have

$$|\sqrt{n+1}-\sqrt{n}-0| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon,$$

which is the definition of convergence. So we've proven that $\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} = 0$. ☐

3. Sequences: Useful Tools

The above method will almost always work; often, however, it can take a lot of work and is ponderous. Consequently, we've developed the following tools to make our lives easier:

(1) **Arithmetic and Sequences**:
- *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
- *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
- *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

(2) **Monotone and Bounded Sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges.

(3) **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l.

(4) **Cauchy sequences** A sequence is Cauchy[1] iff it converges.

[1]We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m, n \geq N$

$$|a_m - a_n| < \epsilon.$$

You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

This next section consists of example of these tools in action:

4. Sequences: Worked Examples

Claim 4.1. *(Arithmetic and Sequences example) The sequence* $a_1 = 1, a_{n+1} = \sqrt{1+a_n^2}$ *does not converge.*

Proof. We proceed by contradiction. Suppose that some limit L of the sequence $\{a_n\}_{n=1}^{\infty}$ exists. Then, examine the limit

$$\lim_{n\to\infty} a_n^2.$$

Because convergent sequences are multiplicative, we know that

$$\lim_{n\to\infty} a_n^2 = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} a_n) = L \cdot L = L^2.$$

However, we can also use the recursive definition of the a_n's to see that

$$\begin{aligned}
\lim_{n\to\infty} a_n^2 &= \lim_{n\to\infty} \left(\sqrt{1+a_{n-1}^2}\right)^2 \\
&= \lim_{n\to\infty} (1+a_{n-1}^2) \\
&= (\lim_{n\to\infty} 1) + (\lim_{n\to\infty} a_{n-1}^2) \\
&= 1 + (\lim_{n\to\infty} a_{n-1}^2) \\
&= 1 + (\lim_{n\to\infty} a_{n-1}) \cdot (\lim_{n\to\infty} a_{n-1}).
\end{aligned}$$

However, we know that $\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n$, because the two sequences are the same (just shifted over one place) and thus have the same behavior at infinity. So we have in fact that

$$\lim_{n\to\infty} a_n^2 = 1 + (\lim_{n\to\infty} a_{n-1}) \cdot (\lim_{n\to\infty} a_{n-1}) = 1 + L^2,$$

and thus that $L^2 = 1 + L^2$, a contradiction. □

Claim 4.2. *(Cauchy sequence example) The sequence*

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

converges.

Proof. To show that this sequence converges, we will use the Cauchy convergence tool, which tells us that sequences converge iff they are Cauchy.

How do we prove that a sequence is Cauchy? As it turns out, we can use a similar blueprint to the methods we used to show that a sequence converges:

- First, examine the quantity $|a_m - a_n|$, and try to come up with a very simple upper bound that depends on m and n and goes to zero. Example bounds we'd love to run into: $\frac{1}{mn}, \frac{1}{n}, \frac{1}{m^4 \log(n)}$. Things that won't work: $\frac{n}{m}$ (if n is really big compared to m, we're doomed!), $\frac{m}{n^{34}}$ (same!), 4.
- Using this upper bound, given $\epsilon > 0$, determine a value of N such that whenever m and $n > N$, our simple bound is less than ϵ.

- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $m, n > N$, $|a_m - a_n| < \epsilon$.

Let's apply the above blueprint, and study $|a_m - a_n|$. Assume that $m > n$ here; the other case will look the exact same (if you flip m and n throughout the proof), so we omit it by symmetry.

$$\begin{aligned}
|a_m - a_n| &= \left| \sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right| \\
&= \sum_{k=n+1}^{m} \frac{1}{k^2} \\
&< \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\
&= \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} \\
&= \sum_{k=n+1}^{m} \frac{1}{k-1} - \sum_{k=n+1}^{m} \frac{1}{k} \\
&= \sum_{k=n}^{m-1} \frac{1}{k} - \sum_{k=n+1}^{m} \frac{1}{k} \\
&= \frac{1}{n} - \frac{1}{m} \\
&< \frac{1}{n} + \frac{1}{m}.
\end{aligned}$$

This looks fairly simple!

Moving onto the second step: given $\epsilon > 0$, we want to force this quantity $\frac{1}{n} + \frac{1}{m} < \epsilon$. How can we do this? Well: if $n, m > N$, we have that $\frac{1}{n} + \frac{1}{m} < \frac{2}{N}$; so it suffices to pick N such that $\frac{2}{N} < \epsilon$.

Thus,we've shown that for any $\epsilon > 0$ we can find a N such that for any $m, n > N$,

$$|a_m - a_n| < \frac{1}{n} + \frac{1}{m} < \frac{2}{N} < \epsilon.$$

But this just means that our sequence is Cauchy! So, because all Cauchy sequences converge, we've proven that our sequence converges. $\square$