MATH 8, SECTION 1, WEEK 3 - RECITATION NOTES

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Abstract. These are the notes from Monday, Oct. 11th's lecture, where we finished our discussion of the squeeze and monotone convergence theorems, and started exploring series.

1. Random Question

Question 1.1. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers, chosen so that the sums $\sum_{n=1}^{\infty} \frac{1}{a_n}$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ both diverge. Does the sum*

$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n}$$

also have to diverge?

2. Sequence Tools, Cont.

Last time, we ended lecture halfway through our examples of various tools we have for studying sequences. Specifically, we had the following list:

(1) **Arithmetic and Sequences**:
- *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
- *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
- *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

(2) **Monotone and Bounded Sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges.

(3) **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l.

(4) **Cauchy sequences** A sequence is Cauchy iff it converges.

Last time, we studied examples that used the properties of arithmetic and sequences, and used the Cauchy property to study that certain sequences converge; in the next two examples, we will look at how to use the two remaining tools (the monotone convergence theorem and squeeze theorem) to study convergence.

Lemma 2.1. *(Monotone convergence theorem example) If c is a real number strictly larger than 1, then*

$$\lim_{n\to\infty} c^{1/n} = 1.$$

Proof. First, notice that for any positive real number d, we have the following chain of equivalent statements:

$$\begin{aligned} & 1 < d \\ \Leftrightarrow & d < d^2 \\ \Leftrightarrow & d^2 < d^3 \\ & \vdots \\ \Leftrightarrow & d^{n-1} < d^n. \end{aligned}$$

In other words, we know that the truth of any of these statements is equivalent to any of the other. Consequently, for any $d \in \mathbb{R}, d > 0$, we have either that all of these statements are true: i.e. that

$$1 < d < d^2 < \ldots < d^{n-1} < d^n,$$

or that all of these statements are false – i.e. that

$$1 \geq d \geq d^2 \geq \ldots \geq d^{n-1} \geq d^n.$$

Let $d = c^{1/n}$. Because $c > 0$, we have that $c^{1/n}$ is positive (as it's defined as the unique real solution to the equation $x^n = c$, and for c positive this equation has only positive solutions.) So, by the above analysis, there are two possibilites: either

$$1 \geq c^{1/n} \geq c^{2/n} \geq \ldots \geq c^{(n-1)/n} \geq c;$$

i.e. $1 \geq c$, which is a contradiction to our claim that $c > 1$; or

$$1 < c^{1/n} < c^{2/n} < \ldots < c^{(n-1)/n} < c.$$

This tells us two things: first, that

$$1 < c^{1/n}, \forall n,$$

and thus that the $c^{1/n}$'s are bounded from below.

Secondly, if we take the string

$$1 < d < d^2 < \ldots < d^{n-1} < d^n$$

again and set $d = c^{1/(n(n-1))}$, we have

$$\begin{aligned} & c^{(n-1)/(n(n-1))} < c^{n/(n(n-1))} \\ \Rightarrow & c^{1/n} < c^{1/(n-1)}. \end{aligned}$$

So the $c^{1/n}$'s are a decreasing sequence; consequently, by the monotone convergence theorem, they have a limit!

We claim finally that it is 1. We know that it cannot be anything less than 1, because 1 is a lower bound. Take any number L greater than 1; then, because $\lim_{n\to\infty} L^n = \infty$, there is some value of N for which L^N is eventually greater than c – i.e. there is some N such that $L > c^{1/N}$.

However, the $c^{1/n}$'s are decreasing! So this means that $|L - c^{1/n}| > |L - c^{1/N}|$ for any $n > N$, where $|L - c^{1/N}|$ is some fixed nonzero positive constant. But this means that L cannot be a limit of our sequence. Thus, by process of elimination, we know that

$$\lim_{n\to\infty} c^{1/n} = 1.$$

□

Lemma 2.2. *(Squeeze theorem example) For any two positive real numbers x, y,*

$$\lim_{n\to\infty} (x^n + y^n)^{1/n} = \max(x, y).$$

Proof. The expression $(x^n + y^n)^{1/n}$ in the limit above is, at first glance, a rather complicated thing to deal with; algebraically, it doesn't look like something that will simplify nicely, and we don't have many other methods for dealing with such complicated expressions. Noticing this "complicatedness" is usually the first step in realizing that you should pursue a squeeze-theorem proof: often, when you run into a sequence that you cannot directly analyze, it's a sign that you *can* study it by bounding it with simpler sequences.

Specifically: in this example, suppose that $x \geq y$ without any loss of generality (as one of our two numbers has to be bigger, and the expression above is symmetric with respect to x and y.) Then, for any n, we have the following bounds:

$$(x^n)^{1/n} \leq (x^n + y^n)^{1/n} \leq (x^n + x^n)^{1/n},$$

where for the left bound we used the observation that $y^n \geq 0$, and for the right bound we used the observation that $x^n \geq y^n$.

These quantities are much easier to study: the left hand side is trivially

$$\lim_{n\to\infty} (x^n)^{1/n} = \lim_{n\to\infty} x = x,$$

and the right hand side is just

$$\lim_{n\to\infty} (2x^n)^{1/n} = \lim_{n\to\infty} 2^{1/n} \cdot x = \left(\lim_{n\to\infty} 2^{1/n}\right) \cdot \left(\lim_{n\to\infty} x\right) = 1 \cdot x = x,$$

by our earlier example.

So: we have exhibited a pair of sequences that bound $(x^n + y^n)^{1/n}$ from the left and right, and both of which go to $\max(x, y)$; thus, by the squeeze theorem, we know that

$$\lim_{n\to\infty} (x^n + y^n)^{1/n} = \max(x, y)$$

as well. □

3. Series: Introduction

Definition 3.1. A sequence is called **summable** if the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums

$$s_n := a_1 + \ldots a_n$$

converges. If it does, we then call the limit of this sequence the **sum** of the a_n, and denote this quantity by writing

$$\sum_{n=1}^{\infty} a_n.$$

We will often denote such infinite sums as **series**.

We've already encountered a series in this class; specifically, we studied the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ as an example of when to use the Cauchy criterion on Friday of week 2. There, we studied this series simply by treating it as a sequence of its partial sums.

Consequently, we might hope that we can transfer a lot of our earlier intuition about sequences to some ideas about series! However, as it turns out, this is a rather hard thing to do: series are, in some senses, much more delicate things than sequences. For example, we can have series that are (on one hand) made out of very small terms that go to zero, and yet (on the other hand) their sums can explode to infinity:

Lemma 3.2. *If* $\{a_n\}_{n=1}^{\infty}$ *is the sequence defined by*

$$a_n = \frac{1}{2^k}, \text{ for } n \in [2^k, 2^{k+1} - 1],$$

then we simultaneously have the following two conditions:

- $\lim_{n\to\infty} a_n = 0$.
- *The series* $\sum_{n=1}^{\infty} a_n$ *does not converge.*

Proof. As the a_n's are decreasing and $a_{2^k} = \frac{1}{2^k}$, it's clear that they converge to 0. However, at the same time, we have that

$$\begin{aligned} a_1 &= 1, \\ a_2 = a_3 &= \frac{1}{2}, \\ a_4 = a_5 = a_6 = a_7 &= \frac{1}{4}, \\ a_8 = \ldots a_{15} &= \frac{1}{8}, \\ &\vdots \end{aligned}$$

and thus that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= (a_1) + (a_2 + a_3) + (a_4 + \ldots + a_7) + (a_8 + \ldots + a_{15}) + \ldots \\ &= 1 + 1 + 1 + 1 + \ldots, \end{aligned}$$

which clearly fails to converge. □

As we saw in the above example, it can be somewhat difficult to simply "eyeball" a sequence and tell if its associated series will converge; this is in contrast to the situation with determining where a sequence would converge to, which we could often "guess" just by looking at it. Series are strange beasts, as we will illustrate further in the next section:

4. Can You Rearrange Terms in a Series?

By definition, the infinte sum $\sum_{n=1}^{\infty} a_n$ denotes the limit

$$\lim_{n\to\infty} a_1 + a_2 + \dots a_n.$$

In the above sum, we assume that when we take this limit, we are always adding up the a_n's "in order" – i.e. we don't look at the limit

$$\lim_{n\to\infty} a_1 + a_2 + a_4 + a_3 + a_6 + a_8 + a_5 + a_{10} + a_{12} + \dots a_{2n-1} + a_{4n-2} + a_{4n},$$

where we're adding up one odd term for every two even terms. Why don't we? After all, in the case of finite sums, the order of addition doesn't matter at all: e.g. $1 + 2 + 3 = 2 + 1 + 3 = 3 + 2 + 1$, regardless of how you add it up. So: does this extend to the case of infinte sums?

To partially answer this question, consider the sequence

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots$$

We first make the following claim about this sequence:

Lemma 4.1. *The series*

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}$$

converges to some positive number $L > 0$.

Proof. So: we seek to show that the sequence of the partial sums

$$\sum_{k=1}^{n} \frac{(1)^{k+1}}{k}$$

converges to some positive number. To do this: examine the partial sums where n is even. Each of these sums is of the form

$$\sum_{k=1}^{n} \frac{(1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{n-1} - \frac{1}{n}.$$

Grouping terms in pairs, we have in fact that this sum is just

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(1)^{k+1}}{k} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{n-1} - \frac{1}{n} \\
&= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{5\cdot 6}{+} \dots + \frac{1}{(n-1)(n)} \\
&\leq \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \frac{1}{(n-1)^2} \\
&\leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},
\end{aligned}$$

and thus that these partial sums are all bounded above by $\frac{\pi^2}{6}$.

As well, the difference between $\sum_{k=1}^{n} \frac{(1)^{k+1}}{k}$ and $\sum_{k=1}^{n+2} \frac{(1)^{k+1}}{k}$ is just the term

$$\frac{1}{n+1} - \frac{1}{n+2},$$

which is strictly positive; so we have that the sequence of even partial sums is (1) bounded above and (2) increasing! So it converges to some limit $L > 0$.

Thus, we know that the sequence

$$(\ddagger) \qquad \sum_{k=1}^{0} \frac{(1)^{k+1}}{k}, \sum_{k=1}^{0} \frac{(1)^{k+1}}{k}, \sum_{k=1}^{2} \frac{(1)^{k+1}}{k}, \sum_{k=1}^{2} \frac{(1)^{k+1}}{k}, \sum_{k=1}^{4} \frac{(1)^{k+1}}{k}, \sum_{k=1}^{4} \frac{(1)^{k+1}}{k} \cdots$$

where we just repeat each even sum twice, must also converge to L. As well, we know that the sequence

$$(\star) \qquad 0, 1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0, \ldots$$

converges to 0; so, because we can perform arithmetic on limits, we have in fact that the sequence formed by summing these two sequences ($\ddagger$) and ($\star$) must also converge.

But the sequence formed by adding ($\ddagger$) and ($\star$) is just the sequence of partial sums of

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n};$$

thus, this series must converge to $L + 0 = L$, as claimed. $\square$

So: why do we mention this series at all? Because, as it turns out, this sequence is an excellent candidate for showing why you **cannot** arbitrarily rearrange terms in a series! For, if you could, we would be able to write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots \\ &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} \ldots\end{aligned}$$

But, if we group terms together in the above rearrangement, we would have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} \ldots \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} \ldots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} \ldots \\ &= \frac{1}{2} \cdot \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots\right) \\ &= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}.\end{aligned}$$

Thus, we have that this sum must be 0, as 0 is the only number that's equal to half of itself. But we just showed that the limit of these partial sums is strictly positive! Thus, we have a contradiction; consequently, we must have that – in some cases, at least – rearranging the terms of a series can completely change what it converges to.

Crazy, right?