MATH 8, SECTION 1, WEEK 3 - RECITATION NOTES

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Abstract. These are the notes from Wednesday, Oct. 13th's lecture, where we studied different methods for analying series and determining whether they converge.

1. Random Question

Question 1.1. *Show that any rational number can be written as a finite sum of distinct numbers of the form $1/n$.*

For an idea on how to approach this question, consider the following algorithm for breaking up $\frac{29}{24}$ into fractions of the form $1/n$: because

$$\frac{29}{24} - \frac{1}{2} = \frac{17}{24}$$
$$\frac{17}{24} - \frac{1}{3} = \frac{9}{24}$$
$$\frac{9}{24} - \frac{1}{4} = \frac{3}{24}$$
$$\frac{3}{24} < \frac{1}{5}$$
$$\frac{3}{24} < \frac{1}{6}$$
$$\frac{3}{24} < \frac{1}{7}$$
$$\frac{3}{24} - \frac{1}{8} = 0,$$

we have that $\frac{29}{24}$ can be written as $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{8}$.

How can you make this into an explicit algorithm that will always work?

2. Series: Some Useful Theorems

In our last class, we introduced the idea of "series," and studied a pair of examples. In doing so, we saw that working with series is a rather tricky thing to do: using only the definition of a series as a limit of partial sums, we had to do a lot of work to show that something as simple as the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converged.

Motivated by this, we've introduced in class a number of useful and powerful theorems, to make our calculations easier. We list them here:

(1) **Comparison Test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then the following statement is true:

$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

When to use this test: when you're looking at something fairly complicated that either (1) you can bound above by something simple that converges, like $\sum 1/n^2$, or (2) that you can bound below by something simple that diverges, like $\sum 1/n$.

(2) **Limit Comparison Test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0,$$

then the following statement is true:

$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Leftrightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

When to use this test: whenever you see something really complicated; so, mostly, in similar situations to the normal comparison test. The advantage to the limit comparison test is that you don't need your terms to always be bigger or smaller; so long as they look the same in the limit, you can use the limit comparison test. Really useful for reducing complicated polynomial expressions to their dominant terms.

(3) **Alternating Series Test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that
- $\lim_{n\to\infty} a_n = 0$ monotonically, and
- the a_n's alternate in sign, then

the series $\sum_{n=1}^{\infty} a_n$ converges. When to use this test: when you have an alternating series.

(4) **Ratio Test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

then we have the following three possibilities:
- If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then we have no idea; it could either converge or diverge.

When to use this test: when you have something that is growing kind of like a geometric series: so when you have terms like 2^n or $n!$.

(5) **Root Test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = r,$$

then we have the following three possibilities:
- If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then we have no idea; it could either converge or diverge.

When to use this test: mostly, in similar situations to the ratio test. Basically, if the ratio test fails, there's a small chance that this will work instead.

3. Series: Examples

To illustrate the use of these theorems, we provide in this section a series of useful examples:

Lemma 3.1. *(Comparison test) If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of positive numbers, with the property that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ both converge, the sum*

$$\sum_{n=1}^{\infty} a_n b_n$$

must also converge.

Proof. To see why, simply note the following inequality: because any number squared is a positive number, we have that

$$\begin{aligned} & 0 \leq (a-b)^2 \\ \Rightarrow \qquad & 0 \leq a^2 + b^2 - 2ab \\ \Rightarrow \qquad & 2ab \leq a^2 + b^2. \end{aligned}$$

Specifically, we have that for any n, $a_n b_n \leq a_n^2 + b_n^2$. But we know that the series $\sum_{n=1}^{\infty} a_n^2 + b_n^2$ converges, by adding both sums together; thus, by the comparison test, we know that this forces

$$\sum_{n=1}^{\infty} a_n b_n$$

to converge as well. □

Lemma 3.2. *If the series $\sum_{n=1}^{\infty} a_n$ converges and all of the a_n's are positive, then the series*

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

converges as well.

Proof. This is simply a special case of our earlier question, if we plug in the two sequences $\{\sqrt{a_n}\}_{n=1}^{\infty}$ and $\{1/n\}_{n=1}^{\infty}$ into our earlier proof. □

Lemma 3.3. *(Ratio test; alternately, limit comparison test + root test) The series*

$$\sum_{n=1}^{\infty} \frac{(n+3)^2}{3^n}$$

converges.

Proof. There are two ways to study this series: the ratio-test way (motivated by the 3^n in the denominator), and the hard way. We present both, to motivate how different methods can lead you to the same proof:

(Ratio test:) Examine the quantity a_{n+1}/a_n:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1+3)^2}{3^{n+1}}}{\frac{(n+3)^2}{3^n}}\\ &= \frac{(n+4)^2 3^n}{(n+3)^2 3^{n+1}}\\ &= \left(\frac{n+4}{n+3}\right)^2 \cdot \frac{1}{3}\\ \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{n+4}{n+3}\right)^2 \cdot \frac{1}{3} = \frac{1}{3};\end{aligned}$$

because this limit exists and is less than 1, we know that our series converges.

(Limit comparison test, comparison test, and root test:) So, because the limit

$$\lim_{n\to\infty} \frac{\frac{(n+3)^2}{3^n}}{\frac{n^2}{3^n}} = \lim_{n\to\infty} \left(\frac{n+3}{n}\right)^2 = 1,$$

the limit comparison test tells us that

$$\left(\sum_{n=1}^{\infty} \frac{(n+3)^2}{3^n} \text{ converges}\right) \Leftrightarrow \left(\sum_{n=1}^{\infty} \frac{n^2}{3^n} \text{ converges}\right).$$

So: that simplifies our polynomial some. But it's not yet simple enough: so what can we do? Well: we know that for any $n \geq 2$, $n^2 \leq 2^n$; so we can use the normal comparison test, which says that

$$\left(\sum_{n=1}^{\infty} \frac{2^n}{3^n} \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} \frac{n^2}{3^n} \text{ converges}\right).$$

But this series is pretty simple! If we remember our geometric series, the left hand side in fact sums to 2; if you forget that, however, you can just apply the root or ratio test to get that, because

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{3}\right)^n} = \frac{2}{3} < 1,$$

the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ converges, and thus (by our earlier work) our original series $\sum_{n=1}^{\infty} \frac{(n+3)^2}{3^n}$ converges as well. □

Lemma 3.4. *(Ratio test) The series*

$$\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^{n+1}}$$

converges.

Proof. Motivated by the presence of both a $n!$ and a 2^n, we try the ratio test:

$$\begin{aligned}\frac{a_n}{a_{n-1}} &= \frac{\frac{2^n \cdot n!}{n^{n+1}}}{\frac{2^{n-1}\cdot(n-1)!}{(n-1)^n}} \\ &= \frac{2^n \cdot n! \cdot (n-1)^n}{2^{n-1}\cdot(n-1)!\cdot n^{n+1}} \\ &= \frac{2\cdot n\cdot(n-1)^n}{n^{n+1}} \\ &= \frac{2\cdot(n-1)^n}{n^n} \\ &= 2\cdot\left(\frac{n-1}{n}\right)^n \\ &= 2\cdot\left(1-\frac{1}{n}\right)^n\end{aligned}$$

Here, we need one bit of knowledge that you may not have encountered before: the limit

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e},$$

the mathematical constant. (Historically, I'm pretty certain that that this is how e was defined; so feel free to take it as a definition of e itself.)

Basically: the relevant bit of information we have here is that $\frac{2}{e}$ is less than 1. So the ratio test tells us that this series converges! □

Lemma 3.5. *The series*

$$\sum_{n=1}^{\infty}\frac{c^n\cdot n!}{n^n}$$

converges if $c < e$, and diverges if $c > e$.

Proof. If we retrace our original proof, swapping in c for 2 only means that at the end, we're looking at the quantity $\frac{c}{e}$ instead of $\frac{2}{e}$. So, if $c < e$, it converges, and if $c > e$, it diverges, again by the ratio test! □