MATH 8, SECTION 1, WEEK 4 - RECITATION NOTES

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Abstract. These are the notes from Monday, Oct. 18th's lecture, where we started to discuss the ideas of limits and continuity.

1. Random Question

Question 1.1. *So, in $\mathbb{R}^2$, you can draw at most 6 equilateral triangles around a given point; this is a simple consequence of the internal angle of a equilateral triangle being 60°. A natural generalization of the above question, then, is the following: in $\mathbb{R}^3$, what is the maximum number of regular tetrahedra can you fit around a given point?*

2. Continuity: Definitions

Definition 2.1. If $f : X \to Y$ is a function between two subsets X, Y of $\mathbb{R}$, we say that

$$\lim_{x \to a} f(x) = L$$

if and only if

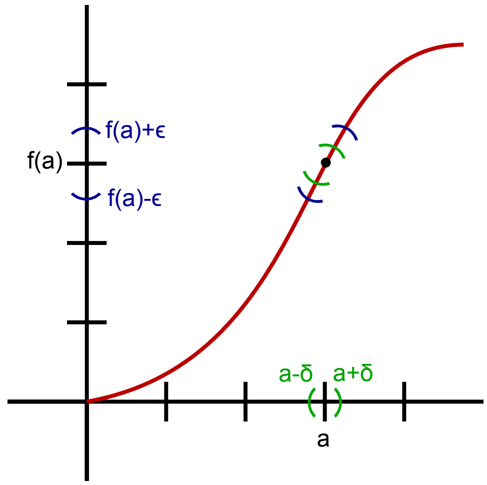
(1) (vague:) as x approaches a, $f(x)$ approaches L.
(2) (precise; wordy:) for any distance $\epsilon > 0$, there is some neighborhood $\delta > 0$ of a such that whenever $x \in X$ is within δ of a, $f(x)$ is within ϵ of L.
(3) (precise; symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

Definition 2.2. A function $f : X \to Y$ is said to be **continuous** at some point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a).$$

Somewhat strange definitions, right? At least, the two "rigorous" definitions are somewhat strange: how do these epsilons and deltas connect with the rather simple concept of "as x approaches a, $f(x)$ approaches $f(a)$"? To see this a bit better, consider the following image:



This graph shows pictorially what's going on in our "rigorous" definition of limits and continuity: essentially, to rigorously say that "as x approaches a, $f(x)$ approaches $f(a)$", we are saying that

- for any distance ϵ around $f(a)$ that we'd like to keep our function,
- there is a neighborhood $(a-\delta, a+\delta)$ around a such that
- if f takes only values within this neighborhood $(a-\delta, a+\delta)$, it stays within ϵ of $f(a)$.

Basically, what this definition says is that if you pick values of x sufficiently close to a, the resulting $f(x)$'s will be as close as you want to be to $f(a)$ – i.e. that "as x approaches a, $f(x)$ approaches $f(a)$."

This, hopefully, illustrates what our definition is trying to capture – a concrete notion of something like convergence for functions, instead of sequences. So: how can we prove that a function f has some given limit L? Motivated by this analogy to sequences, we have the following blueprint for a proof-from-the-definitions that $\lim_{x\to a} f(x) = L$:

(1) First, examine the quantity
$$|f(x) - L|.$$
Specifically, try to find a simple upper bound for this quantity that depends only on $|x-a|$, and goes to 0 as x goes to a – something like $|x-a| \cdot$ (constants), or $|x-a|^3 \cdot$ (bounded functions, like $\sin(x)$).

(2) Using this simple upper bound, for any $\epsilon > 0$, choose a value of δ such that whenever $|x-a| < \delta$, your simple upper bound $|x-a| \cdot$ (constants) is $< \epsilon$. Often, you'll define δ to be $\epsilon/$(constants), or somesuch thing.

(3) Plug in the definition of the limit: for any $\epsilon > 0$, we've found a δ such that whenever $|x-a| < \delta$, we have
$$|f(x) - L| < (\text{simple upper bound depending on } |x-a|) < \epsilon.$$
Thus, we've proven that $\lim_{x\to a} f(x) = L$, as claimed.

The following example ought to illustrate what we're talking about here:

3. Continuity: An Example

Lemma 3.1. *The function $\frac{1}{x^2}$ is continuous at every point $a \neq 0$.*

Proof. We want to prove that $\lim_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$, for any $a \neq 0$.

We proceed according to our blueprint:

(1) First, we examine the quantity $\left|\frac{1}{x^2} - \frac{1}{a^2}\right|$:

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{a^2}\right| &= \left|\frac{a^2}{a^2x^2} - \frac{x^2}{a^2x^2}\right| \\
&= \left|\frac{a^2 - x^2}{a^2x^2}\right| \\
&= \left|\frac{(a-x)(a+x)}{a^2x^2}\right| \\
&= |a-x| \cdot \left|\frac{(a+x)}{a^2x^2}\right| \\
&= |x-a| \cdot \left|\frac{(a+x)}{a^2x^2}\right|.
\end{aligned}$$

By algebraic simplification, we've broken our expression into two parts: one of which is $|x-a|$, and the other of which is... something. We'd like to get rid of this extra part $\left|\frac{(a+x)}{a^2x^2}\right|$; so, how do we do this? We cannot just say that this quantity is bounded; indeed, for very small values of x, this explodes off to infinity.

But for values of x rather close to a, because $a \neq 0$, this is bounded! In fact, if we have values of x such that x is within $a/2$ of a, we have

$$\begin{aligned}
\left|\frac{(a+x)}{a^2x^2}\right| &\leq \left|\frac{(a+(3a/2))}{a^2x^2}\right| \\
&\leq \left|\frac{(a+(3a/2))}{a^2(a/2)^2}\right| \\
&= \left|\frac{10}{a^3}\right|
\end{aligned}$$

,

which is some nicely bounded constant. So, when we pick our δ, if we just make sure that $\delta < a/2$, we know that we have this quite simple and excellent upper bound

$$|f(x) - f(a)| \leq |x-a| \cdot \left|\frac{10}{a^3}\right|.$$

(2) We have a simple upper bound! Our next step then proceeds as follows: for any $\epsilon > 0$, we want to pick a $\delta > 0$ such that if $|x-a| < \delta$,

$$|x-a| \cdot \left|\frac{10}{a^3}\right| < \epsilon.$$

But this is rather simple: if we want this to happen, we merely need to pick δ so that $\delta < a/2$ (so we get to use our nice simple upper bound,) and also so that $\delta < \epsilon / \frac{10}{|a|^3}$. Explicitly, we can pick $\delta < \min\left(a/2, \epsilon/\frac{10}{|a|^3}\right)$.

(3) Thus, for any $\epsilon > 0$, we've found a $\delta > 0$ such that whenever $|x - a| < \delta$, we have

$$|x-a| \cdot \left|\frac{10}{a^3}\right| < \epsilon.$$

Thus, $\lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2}$ for any $a \neq 0$, as claimed.

□

4. Continuity: Three Useful Tools

Limits and continuity are wonderfully useful concepts, but working with them straight from the definitions – as we saw above – can be somewhat ponderous. As a result, we have developed a number of useful tools and theorems to allow us to prove that certain limits exist without going through the definition every time: we present three such tools, and examples for each, here.

Theorem 4.1. *(Squeeze theorem:) If f, g, h are functions defined on some interval $I \setminus \{a\}$*[1] *such that*

$$f(x) \leq g(x) \leq h(x), \forall x \in I \setminus \{a\},$$
$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x),$$

then $\lim_{x \to a} g(x)$ exists, and is equal to the other two limits $\lim_{x \to a} f(x), \lim_{x \to a} h(x)$.

Example 4.2.

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$

Proof. So: for all $x \in \mathbb{R}, x \neq 0$, we have that

$$-1 \leq \sin(1/x) \leq 1$$
$$\Rightarrow -x^2 \leq x^2 \sin(1/x) \leq x^2;$$

thus, by the squeeze theorem, as the limit as $x \to 0$ of both $-x^2$ and x^2 is 0,

$$\lim_{x \to 0} x^2 \sin(1/x) = 0$$

as well. □

[1]The set $X \setminus Y$ is simply the set formed by taking all of the elements in X that are not elements in Y. The symbol $\setminus$, in this context, is called "set-minus", and denotes the idea of "taking away" one set from another.

Theorem 4.3. *(Limits and arithmetic): if f, g are a pair of functions such that $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist, then we have the following equalities:*

$$\lim_{x\to a}(\alpha f(x) + \beta g(x)) = \alpha\left(\lim_{x\to a} f(x)\right) + \beta\left(\lim_{x\to a} g(x)\right)$$
$$\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$$
$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \left(\lim_{x\to a} f(x)\right)/\left(\lim_{x\to a} g(x)\right), \text{ if } \lim_{x\to a} g(x) \neq 0.$$

Corollary 4.4. *Every polynomial is continuous everywhere.*

Proof. To start, we know that the functions $f(x) = x$ and $f(x) = 1$ are trivially continuous. By multiplying these functions together and scaling by constant factors, we can create any polynomial; thus, by the above theorem, we know that any polynomial must be continuous, as we can create it from continuous things through arithmetical operations. □

Theorem 4.5. *(Limits and composition): if $f : Y \to Z$ is a function such that $\lim_{y\to a} f(x) = L$, and $g : X \to Y$ is a function such that $\lim_{x\to b} g(x) = a$, then*

$$\lim_{x\to b} f(g(x)) = L.$$

Specifically, if both functions are continuous, their composition is continuous.

Example 4.6.

$$\lim_{x\to a} \sin(1/x^2) = \sin(1/a^2),$$

if $a \neq 0$.

Proof. By our work earlier in this lecture, $1/x^2$ is continuous at any value of $a \neq 0$, and from class $\sin(x)$ is continuous everywhere: thus, we have that their composition, $\sin(1/a^2)$, is continuous wherever $x \neq 0$. Thus,

$$\lim_{x\to a} \sin(1/x^2) = \sin(1/a^2),$$

as claimed. □