MATH 8, SECTION 1, WEEK 4 - RECITATION NOTES

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Abstract. These are the notes from Wednesday, Oct. 20th's lecture, where we studied several different kinds of discontinuous functions.

1. Random Question

Question 1.1. *Can you find a function $f : \mathbb{R} \to \mathbb{R}$ that's*

- *continuous at every rational point $q \in \mathbb{Q}$, but*
- *discontinuous at every irrational point $a \in \mathbb{R} \setminus \mathbb{Q}$?*

2. Discontinuity Proofs: A Lemma and a Blueprint

How do we show a function is discontinuous? Specifically: in our last class, we described a "blueprint" for showing that a given function was continuous at a point. Can we do the same for the concept of discontinuity?

As it turns out, we can! Specifically, we have the following remarkably useful lemma, proved in Dr. Ramakrishnan's class on Wednesday:

Lemma 2.1. *For any function $f : X \to Y$, we know that $\lim_{x\to a} f(x) \neq L$ iff there is some sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:*

- $\lim_{n\to\infty} a_n = L$*, and*
- $\lim_{n\to\infty} f(a_n) \neq L$*, and*

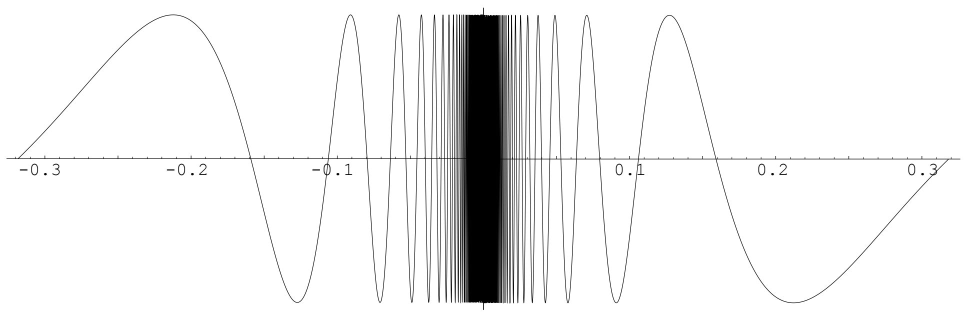
This lemma makes proving that a function f is discontinuous at some point a remarkably easy: all we have to do is find a sequence $\{a_n\}_{n=1}^{\infty}$ that converges to a on which the values $f(a_n)$ fail to converge to $f(a)$. Basically, it allows us to work in the world of sequences instead of that of continuity; a change that makes a lot of our calculations easier to make.

The following three examples should help illustrate our methods here:

3. Three Discontinuous Functions

Lemma 3.1. *The function $\sin(1/x)$ has no defined limit at 0.*

Proof. (This proof has been redacted, as it's apparently on the problem set this week! Oops.) So: before we start, consider the graph of $\sin(1/x)$:



Visual inspection of this graph makes it clear that $\sin(1/x)$ cannot have a limit as x approaches 0; but let's rigorously prove this using our lemma, so we have an idea of how to do this in general.—

So: we know that $\sin\left(\frac{4k+1}{2}\pi\right) = 1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+1)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+1)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+1)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+1}{2}\pi\right) = \lim_{k\to\infty} 1 = 1$,

our lemma says that if $\sin(1/x)$ has a limit at 0, it must be 1.

However: we also know that $\sin\left(\frac{4k+3}{2}\pi\right) = -1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+3)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+3)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+3)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+3}{2}\pi\right) = \lim_{k\to\infty} -1 = -1$,

our lemma **also** says that if $\sin(1/x)$ has a limit at 0, it must be -1. Thus, because $-1 \neq 1$, we have that the limit $\lim_{x\to 0} \sin(1/x)$ cannot exist, as claimed.

$\square$

Question 3.2. *Can you find a function $f : \mathbb{R} \to \mathbb{R}$ that's discontinuous everywhere?*

Proof. (Again, as this question is apparently #3 on your problem set this week, this example has been redacted as well. Sorry!)

As it turns out: yes! Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We claim that this function is discontinuous at every point $a \in \mathbb{R}$. To see this: first, recall that both the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are **dense** in the real numbers – in other words, for any two real numbers x, y such that $x < y$, we have that there is a rational number q and irrational number β such that $x < q < y$ and $x < \beta < y$.

As a corollary, this means that for any number $a \in \mathbb{R}$, we can find a rational number q_n and an irrational number β_n such that

$$a - \frac{1}{n} < q_n < a, \text{ and } a - \frac{1}{n} < \beta_n < a.$$

Look at the sequences $\{q_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ formed by the numbers we picked above. Because both the sequences $\left\{a - \frac{1}{n}\right\}_{n=1}^{\infty}$ and $\{a\}_{n=1}^{\infty}$ converge to a, the squeeze theorem then tells us that the sequences $\{q_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ must both converge to a as well. However, by definition, we know that

$$\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 1 = 1, \text{ and } \lim_{n\to\infty} f(\beta_n) = \lim_{n\to\infty} 0 = 0.$$

Thus, by our lemma, if the limit $\lim_{x\to a} f(x)$ were to ever exist, for any a, it would have to simultaneously be equal to 0 and 1. As this is impossible, we conclude that the limit $\lim_{x\to a} f(x)$ cannot exist, for any $a \in \mathbb{R}$. □

Question 3.3. *Can you find a function $f : \mathbb{R} \to \mathbb{R}$ that's*

- *continuous at every irrational point $a \in \mathbb{R} \setminus \mathbb{Q}$, but*
- *discontinuous at every rational point $q \in \mathbb{Q}$?*

Proof. (Un-redacted math! yay!) As it turns out: yes! Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ and } GCD(p, q) = 1; \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}; \\ 1, & x = 0. \end{cases}$$

Take any rational point $\frac{a}{b} \in \mathbb{Q}$; we claim that our function is discontinuous at $\frac{a}{b}$. To see this, we proceed in a similar fashion to our earlier example: specifically, using the densty of the irrational numbers in $\mathbb{R}$, choose a sequence $\{\beta_n\}_{n=1}^{\infty}$ of irrational numbers such that

$$\beta_n \in \mathbb{R} \setminus \mathbb{Q}, \text{ and } \frac{a}{b} - \frac{1}{n} < \beta_n < \frac{a}{b}.$$

By construction, this sequence converges to $\frac{a}{b}$. However, we have that

$$\lim_{n\to\infty} f(\beta_n) = \lim_{n\to\infty} 0 = 0;$$

thus, by our lemma, if a limit exists at $\frac{a}{b}$, it must be 0. However, if we look at our function f at $\frac{a}{b}$, we can see that $f\left(\frac{a}{b}\right) = \frac{1}{b}$ if $\frac{a}{b} \neq 0$, and 1 if $\frac{a}{b} = 0$; as neither of these values is equal to 0, we know that our function must be discontinuous at 0.

That finishes half of our proof; the remaining half, then, is in proving that for any irrational number $a \in \mathbb{R}$,

$$\lim_{x\to a} f(x) = 0.$$

To see this: start by assuming that $a \in [0, 1]$ (at the end of the proof, we'll extend this result to any irrational a.)

Let the sequence $\{q_n\}_{n=1}^{\infty}$ consist of all of the rational numbers in $[0, 1]$ ordered by the size of their denominators: i.e.

$$\{q_n\}_{n=1}^{\infty} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots\right\}$$

Because a is irrational, we know that the distance $|a - q_n| \neq 0$, for any n: so, define $\gamma_n = |a - q_n|$ to be that distance, for every n, and let

$$\{\gamma_n\}_{n=1}^{\infty} = \{\gamma_1, \gamma_2, \gamma_3 \ldots\}$$

To prove that our function is continuous at a: pick any $\epsilon > 0$, and choose N such that $\frac{1}{N} < \epsilon$. Then, if we want to keep our function $f(x)$ within at least $\frac{1}{N}$

of 0, we just need to insure that all of our x-values are either irrational or have denominators $\geq N$; i.e. we need to pick values of x such that

$$x \notin \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{1}{N}, \ldots \frac{N-1}{N}\right\}.$$

Let Q_N denote the above collection of all rational numbers in $[0, 1]$ with denominators $\leq N$. Let Γ_N denote the associated set of distances γ_i to this set Q_N: i.e.

$$\Gamma_N = \{\gamma_1, \ldots \gamma_{\text{something}}\}.$$

Γ_N is a finite set: so it has a minimum $\gamma > 0$.

Set $\delta = \min(\gamma, a, 1-a)$: then, for any x with $|x-a| < \delta$, we have that

- $x > 0$, because $\delta \leq a$,
- $1 > x$, because $\delta \leq 1 - a$, and
- if x is rational, the denominator of x is strictly greater than N: this is because $|x-a| < \delta \leq \gamma$, and any rational number in $[0, 1]$ with denominator $\leq N$ is at least γ away from a.

Thus, we either have that x is rational, is in $[0, 1]$, and has denominator strictly greater than N (in which case $|f(x)| < 1/N < \epsilon$), or we have that x is irrational (in which case $|f(x)| = 0 < \epsilon$.) In either case, we have found a δ such that whenever $|x-a| < \delta$, $|f(x) - 0| < \epsilon$; so this function is continuous at any irrational a in $[0, 1]$.

To extend this result to any irrational a: if a is in some interval $[n, n+1]$, simply translate all of the relevant parts in this proof by n. All of our calculations go through as before: thus, we have in fact that our function is continuous at any irrational value a. So we're done!

$\square$