MATH 8, SECTION 1, WEEK 5 - RECITATION NOTES

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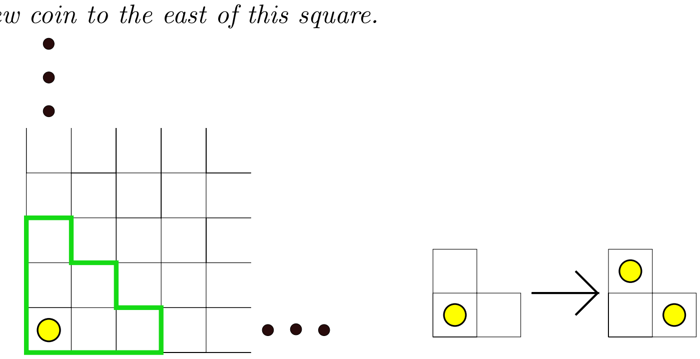
Abstract. These are the notes from Wednesday, Oct. 27th's lecture, where we studied how differentiability and extrema are related; also C^1 functions!

1. Random Question

Question 1.1. *Suppose you have a $\mathbb{N} \times \mathbb{N}$ grid of 1×1 squares. Consider the following game you can play on this board:*

- *Starting configuration: put one coin on the square in the bottom-right-hand corner of our board.*
- *Moves: If we ever have a coin such that the square immediately above this coin and the square immediately to the right of this coin are empty, we can remove this coin from the board, and put one new coin to the north and one new coin to the east of this square.*



Given this setup, the question is this: Is there some finite sequence of moves that can empty the highlighted green region of all its coins?

2. Interpretations of the Derivative

Last class, we defined the derivative as follows:

Definition 2.1. For a function f differentiable at a point a, we say that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}.$$

There are a pair of useful ways to interpret this limit:

- If we think of t as a measurement of time and $f(t)$ as a function that outputs a measurement of distance, $f'(t)$ is measuring the change in distance over the change in time – in other words, the **velocity** of the distance function f. Similarly, $f''(t)$ is measuring the change in velocity over the change in distance: i.e. the **acceleration**.

- If we think of $f(x)$ as a function that outputs the height at location x, $f'(t)$ is measuring the change in height over the change in x-position – in other words, the rise over the run, or the **slope** of the graph of f at x. Similarly, $f''(x)$ can be thought of as measuring the **curvature** of the graph of f at x.

3. Extrema and Derivatives

In class, kind-of-motivated by the above ways to interpret the derivative, we found out that the derivative can be used to classify the **extrema** of various functions, via the following definitions/propositions:

Definition 3.1. A function f has a **critical point** at some point x if either of the two properties hold:

- f is not differentiable, or
- $f'(x) = 0$.

Definition 3.2. A function f has a **local maxima**(resp. **local minima**) at some point a iff there is a neighborhood $(a - \delta, a + \delta)$ around a such that $f(a) \geq f(x)$ (resp. $f(a) \leq f(x)$,) for any $x \in (a - \delta, a + \delta)$.

Proposition 3.3. *If f is a function that has a local minima or maxima at some point t, t is a critical point of f. As a corollary, if f is a continuous function defined on some interval $[a, b]$, f adopts its global minima and maxima at either f's critical points in $[a, b]$, or at the endpoints $\{a, b\}$ themselves.*

Furthermore, if $f''(x)$ is defined and positive at one of these critical points, f adopts a local minima at x; conversely, if $f''(x)$ exists and is negative, f adopts a local maxima at x.

To show how this works, consider the following example:

Question 3.4. *Choose $n \in \mathbb{N}$. Where does the function*

$$f(x) = x^n - xn^n$$

take its local and global minima and maxima in the interval $[-2n, 2n]$?

Proof. First, note that if $n = 1$ our function is identically 0, and thus its local and global minima and maxima are uninteresting. We will focus on $n > 1$ for the rest of the proof.

By the above proposition, we know that f will take on these minima and maxima at its critical points and endpoints. Because f is differentiable everywhere, f's only critical points come at places where $f'(x) = 0$. We examine these points here:

$$\begin{aligned} f'(x) &= nx^{n-1} - n^n = 0 \\ \Leftrightarrow nx^{n-1} &= n^n \\ \Leftrightarrow \ x^{n-1} &= n^{n-1}. \end{aligned}$$

There are two cases, here: if n is odd, its critical points occur at $\pm n$; if n is even, however, its only critical point is at n. In either situation, we have that $f''(x) = n(n-1)x^{n-2}$; thus, we have that $x = n$ is a local minima regardless of whether n is odd or even, while $x = -n$ is a local maxima for n odd.

This accomplished, we can then evaluate our function at these points along with the endpoints, and use this to find its global maxima and minima:

For n odd:

$$\begin{aligned}
f(-2n) &= (-2n)^n - (-2n) \cdot n^n = n^n \left(2n - 2^n\right), \\
f(-n) &= (-n)^n - (-n) \cdot n^n = n^n \left(n - 1\right), \\
f(n) &= (n)^n - (n) \cdot n^n = n^n \left(1 - n\right), \\
f(2n) &= (2n)^n - (2n) \cdot n^n = n^n \left(2^n - 2n\right),
\end{aligned}$$

So: if $n > 2$, we know that $2^n > 2n$; consequently, we have that $f(-2n)$ is the global minima and $f(2n)$ is the global maxima. Because every odd number other than 1 is > 2, we've thus resolved our question of n odd.

For n even:

$$\begin{aligned}
f(-2n) &= (-2n)^n - (-2n) \cdot n^n = n^n \left(2n + 2^n\right), \\
f(n) &= (n)^n - (n) \cdot n^n = n^n \left(1 - n\right), \\
f(2n) &= (2n)^n - (2n) \cdot n^n = n^n \left(2^n - 2n\right).
\end{aligned}$$

For any even value of n, this function has its global maxima at $f(-2n)$ and its global minima at $f(n)$. Thus, we've classified f's local and global minima and maxima for any value of n: so we're done! □

4. C^1- functions

Definition 4.1. A function is called C^1 on some interval $[a, b]$ iff the following holds: f is continuous on $[a, b]$, and f' is also continuous on $[a, b]$. Polynomials, trig functions, and e^x are all C^1 functions.

When people first run into the definition of C^1, they often misstate it as only requiring that f is continuous on $[a, b]$ and merely that $f'(x)$ **exists** on $[a, b]$. This raises the following question: can you have a function that's continuous and has a well-defined derivative, but have that derivative somehow be discontinuous?

As it turns out, the answer is yes! We study this in our last example:

Lemma 4.2. *The function*

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is

- *continuous on all of* $\mathbb{R}$,
- *has a derivative at every point* $a \in \mathbb{R}$, *and yet*
- f' *is not continuous. Specifically,* f' *is not continuous at 0.*

Proof. By bounding $f(x)$ above and below by $\pm x^2$ and applying the squeeze theorem, we can see that f is continuous at 0; at any other point in $\mathbb{R}$, it's the composition/product of continuous functions and is thus continuous. So f is a continuous function on all of $\mathbb{R}$.

Similarly, at any point $a \neq 0$, we know that f is the composition/product of differentiable functions: thus, by applying the chain and product rules, we can see that

$$\begin{aligned} f'(a) &= \left((x^2)' \cdot \sin(1/x) + x^2(\sin(1/x))'\right)\Big|_a \\ &= \left(2x \cdot \sin(1/x) + x^2 \cdot \cos(1/x) \cdot (-1/x^2)\right)\Big|_a \\ &= 2a\sin(1/a) + \cos(1/a), \end{aligned}$$

which is continuous at any point $a \neq 0$ and cannot have a limit as $a \to 0$, as we've shown before. So all we have to do is show that our function is in fact differentiable at a, and we'll have found a function that's continuous and has a derivative on $\mathbb{R}$, and yet doesn't have a continuous derivative!

But this is just a limit calculation, using the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{(0+h) - 0} \\ &= \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} \\ &= \lim_{h \to 0} h \sin(1/h) \qquad\qquad = 0. \end{aligned}$$

Thus f is differentiable at 0, as claimed. $\square$