MATH 8, SECTION 1, WEEK 6 - RECITATION NOTES

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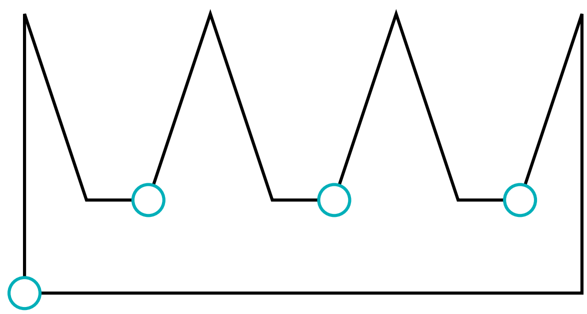
Abstract. These are the notes from Wednesday, Nov. 3rd's lecture, where we calculated the integral of x^p.

1. Random Question

Question 1.1. *Suppose that you have an art gallery that is shaped like some sort of n-polygon, and you want to place cameras with 360°-viewing angles along the vertices of your polygon in such a way that the entire gallery is under surveillance. How many cameras do you need?*



Above: an example 4-camera solution for the above 12-gon art gallery.

2. Integrating x^p

Today's lecture is centered around proving the following claim:

Lemma 2.1. *The function $f(x) = x^p$ is integrable on $[0, b]$ for any $p \in \mathbb{N}$ and $b \in \mathbb{R}^+$. Furthermore, the integral of this function is $\frac{b^{p+1}}{p+1}$.*

Proof. For convenience, we restate one of our definitions of the integral here:

Definition 2.2. A function f is **integrable** on the interval $[a, b]$ if and only if there is a sequence of partitions $\{P_n\}$ such that

$$\lim_{n\to\infty}\left(\sum_{P_n}\sup_{x\in(t_{i-1},t_i)}(f(x))\cdot\text{length}(t_{i-1},t_i)-\sum_{P_n}\inf_{x\in(t_{i-1},t_i)}(x^p)\cdot\text{length}(t_{i-1},t_i)\right)=0.$$

If this happens and the limit can be split over the two sums above, then we write

$$\int_a^b f(x)dx=\lim_{n\to\infty}\left(\sum_{P_n}\sup_{x\in(t_{i-1},t_i)}(f(x))\cdot\text{length}(t_{i-1},t_i)\right)=\lim_{n\to\infty}\left(\sum_{P_n}\inf_{x\in(t_{i-1},t_i)}(f(x))\cdot\text{length}(t_{i-1},t_i)\right),$$

and say that this quantity is the **integral** of $f(x)$ on the interval $[a, b]$.

So: how do we find this sequence of partitions? Well: one trick that will usually work is to simply partition $[0, b]$ into n equal parts – i.e. to consider the partition $0 < \frac{b}{n} < 2\frac{b}{n} < \ldots < n\frac{b}{n} = b.$. Under this partition, we have that the upper-bound sum, ($\sum \sup$), is

$$\begin{aligned}\sum_{P_n} \sup_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i) &= \sum_{k=1}^{n} \sup_{x \in \left(\frac{(k-1)b}{n}, \frac{kb}{n}\right)} (x^p) \cdot \text{length}\left(\frac{(k-1)b}{n}, \frac{kb}{n}\right) \\ &= \sum_{k=1}^{n} \left(\frac{kb}{n}\right)^p \cdot \frac{b}{n} \\ &= \frac{b^{p+1}}{n^{p+1}} \sum_{k=1}^{n} k^p,\end{aligned}$$

and that the lower-bound sum, ($\sum \inf$), is

$$\begin{aligned}\sum_{P_n} \inf_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i) &= \sum_{k=1}^{n} \inf_{x \in \left(\frac{(k-1)b}{n}, \frac{kb}{n}\right)} (x^p) \cdot \text{length}\left(\frac{(k-1)b}{n}, \frac{kb}{n}\right) \\ &= \sum_{k=1}^{n} \left(\frac{(k-1)b}{n}\right)^p \cdot \frac{b}{n} \\ &= \frac{b^{p+1}}{n^{p+1}} \sum_{k=1}^{n} (k-1)^p.\end{aligned}$$

Taking their difference, we have that

$$\begin{aligned}&\lim_{n \to \infty} \left(\sum_{P_n} \sup_{x \in (t_{i-1}, t_i)} (x^p) \cdot \text{length}(t_{i-1}, t_i) - \sum_{P_n} \inf_{x \in (t_{i-1}, t_i)} (x^p) \cdot \text{length}(t_{i-1}, t_i) \right) \\ &= \lim_{n \to \infty} \left(\frac{b^{p+1}}{n^{p+1}} \sum_{k=1}^{n} k^p - \frac{b^{p+1}}{n^{p+1}} \sum_{k=1}^{n} (k-1)^p \right) \\ &= \lim_{n \to \infty} \frac{b^{p+1}}{n^{p+1}} \left(\sum_{k=1}^{n} k^p - \sum_{k=1}^{n} (k-1)^p \right) \\ &= \lim_{n \to \infty} \frac{b^{p+1}}{n^{p+1}} (n^p) \\ &= \lim_{n \to \infty} \frac{b^{p+1}}{n} \\ &= 0.\end{aligned}$$

Thus, by our definition, the function x^p is integrable on $[0, b]$! Furthermore, we know that its integral is in fact given by the limit of either these upper sums or lower sums: i.e. that

$$\int_0^b x^p dx = \lim_{n\to\infty}\left(\sum_{P_n} \sup_{x\in(t_{i-1},t_i)} (x^p)\cdot \text{length}(t_{i-1},t_i)\right) = \lim_{n\to\infty} \frac{b^{p+1}}{n^{p+1}}\sum_{k=1}^n k^p$$
$$= \lim_{n\to\infty}\left(\sum_{P_n} \inf_{x\in(t_{i-1},t_i)} (x^p)\cdot \text{length}(t_{i-1},t_i)\right) = \lim_{n\to\infty} \frac{b^{p+1}}{n^{p+1}}\sum_{k=1}^n (k-1)^p.$$

So: how do we calculate these sums? At first, the answer isn't completely obvious: how can we evaluate these sums $\sum_{k=1}^n k^p$, for any $p \in \mathbb{N}$?

At first, we can try calculating these sums for some sample values of p: for $p = 0$, for example, we have that

$$\sum_{k=1}^n k^p = \sum_{k=1}^n k^0 = \sum_{k=1}^n 1 = n,$$

and thus that

$$\begin{aligned}\int_0^b x^0 dx &= \lim_{n\to\infty} \frac{b^{0+1}}{n^{0+1}}\sum_{k=1}^n k^0\\ &= \lim_{n\to\infty} \frac{b}{n}\cdot n\\ &= b,\end{aligned}$$

which is indeed the integral of $f(x) = x^0 = 1$ from 0 to b.

Similarly, for $p = 1$, we have that

$$\sum_{k=1}^n k^p = \sum_{k=1}^n k^1 = \sum_{k=1}^n k = \frac{(n)(n+1)}{2},$$

by Euler's identity. Thus, we have that

$$\begin{aligned}\int_0^b x^0 dx &= \lim_{n\to\infty} \frac{b^{1+1}}{n^{1+1}}\sum_{k=1}^n k^1\\ &= \lim_{n\to\infty} \frac{b^2}{n^2}\cdot\frac{(n)(n+1)}{2}\\ &= \frac{b^2}{2}\cdot\lim_{n\to\infty}\frac{n^2+n}{n^2}\\ &= \frac{b^2}{2},\end{aligned}$$

which again agrees with our claim that the integral should be $b^{p+1}/(p+1)$.

Using the identity

$$\sum_{k=1}^n k^2 \frac{(n)(n+1)(2n+1)}{6},$$

we can perform a similar calculation for $p = 2$. However, there doesn't seem to be an immediately obvious pattern emerging here; i.e. the sums $\sum_{k=1}^n k^p$ aren't getting any easier to deal with! So: how do we deal with complicated objects whose limits we want to study?

With the squeeze theorem! Specifically, we can do the following:

(1) If we can show that

$$\left(\sum_{P_n} \inf_{x\in(t_{i-1},t_i)} (x^p)\cdot \text{length}(t_{i-1},t_i)\right) \leq \frac{b^{p+1}}{p+1} \leq \left(\sum_{P_n} \sup_{x\in(t_{i-1},t_i)} (x^p)\cdot \text{length}(t_{i-1},t_i)\right)$$

holds for every $n \in \mathbb{N}$, then

(2) the squeeze theorem will tell us that, because

$$\begin{aligned}\int_0^b x^p dx &= \lim_{n\to\infty} \left(\sum_{P_n} \sup_{x\in(t_{i-1},t_i)} (x^p)\cdot \text{length}(t_{i-1},t_i)\right)\\ &= \lim_{n\to\infty} \left(\sum_{P_n} \inf_{x\in(t_{i-1},t_i)} (x^p)\cdot \text{length}(t_{i-1},t_i)\right),\end{aligned}$$

that the middle quantity must also converge to $\int_a^b x^p dx$! I.e, that

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1},$$

which is what we want to prove.

Therefore, it suffices to prove that the claim in (1) holds: i.e. that for any $b > 0$ and $p \in \mathbb{N}$,

$$\frac{b^{p+1}}{n^{p+1}}\sum_{k=1}^{n}(k-1)^p \leq \frac{b^{p+1}}{p+1} \leq \frac{b^{p+1}}{n^{p+1}}\sum_{k=1}^{n} k^p.$$

Dividing through by b^{p+1} and multiplying by n^{p+1}, this becomes the claim

$$\sum_{k=1}^{n}(k-1)^p \leq \frac{n^{p+1}}{p+1} \leq \sum_{k=1}^{n} k^p.$$

How do we prove this? Well: first, let's prove the following useful algebraic identity:

Lemma 2.3. *For any $k \in \mathbb{N}$, we have that*

$$(k-1)^p \leq \frac{k^{p+1}-(k-1)^{p+1}}{p+1} \leq k^p.$$

Proof. First: remember that for any $x < y \in \mathbb{R}$, we have the equality

$$x^{p+1} - y^{p+1} = (x-y)(x^p + x^{p-1}y + x^{p-2}y^2 + \ldots + xy^{p-1} + y^p).$$

If we let $x = k-1$ and $y = k$, this becomes the statement

$$\begin{aligned}k^{p+1} - (k-1)^{p+1} &= (k-(k-1))((k-1)^p + (k-1)^{p-1}k + \ldots + k^p)\\ &= (k-1)^p + (k-1)^{p-1}k + \ldots + k^p.\end{aligned}$$

This right-hand side, however, can be bounded rather nicely from above and below! Specifically, notice that because $(k-1)^p \leq (k-1)^l \cdot k^{p-l} \leq k^p$ for any l, we have

$$(k-1)^p + (k-1)^p + \ldots + (k-1)^p \leq (k-1)^p + (k-1)^{p-1}k + \ldots + k^p \leq k^p + k^p + \ldots + k^p.$$

As there are $p+1$-many terms in the middle part, we can simplify this inequality to the statement

$$(p+1)(k-1)^p \leq (k-1)^p + (k-1)^{p-1}k + \ldots + k^p \leq (p+1)k^p,$$

which becomes, after dividing through by $(p+1)$,

$$(k-1)^p \leq \frac{(k-1)^p + (k-1)^{p-1}k + \ldots + k^p}{p+1} \leq k^p.$$

Plugging in the first equality that we derived, $k^{p+1}-(k-1)^{p+1} = (k-1)^p+\ldots+k^p$, then gives us that

$$(k-1)^p \leq \frac{k^{p+1} - (k-1)^{p+1}}{p+1} \leq k^p,$$

which is what we wanted to prove. □

A trivial consequence of the above lemma is that

$$\sum_{k=1}^{n} (k-1)^p \leq \sum_{k=1}^{n} \frac{k^{p+1} - (k-1)^{p+1}}{p+1} \leq \sum_{k=1}^{n} k^p.$$

Now, to finish our proof, simply notice that the middle sum is telescoping! In other words, that

$$\sum_{k=1}^{n} \frac{k^{p+1} - (k-1)^{p+1}}{p+1} = \frac{n^{p+1}}{p+1}.$$

Consequently, we've proven that

$$\sum_{k=1}^{n} (k-1)^p \leq \frac{n^{p+1}}{p+1} \leq \sum_{k=1}^{n} k^p,$$

and thus (via the algebra done earlier) that the quantity $\frac{b^{p+1}}{p+1}$ always lies between the $(\sum \sup)$ and $(\sum \inf)$ sums. Therefore, by the squeeze theorem, we must have that

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1},$$

which is what we claimed. □