MATH 8, SECTION 1, WEEK 7 - RECITATION NOTES

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Abstract. These are the notes from Monday, Nov. 15th's lecture, where we demonstrated a rather curious substitution.

1. Random Question

Question 1.1. *A **6-sided die** is exactly what you think it is: a cube with six numbers on it, one printed on each face, such that when you roll the die each face has the same probability (1/6) of coming up.*

Can you create a pair of dice A, B with face values in $\mathbb{N}$ such that

- *neither A nor B are equal to the standard die $\{1, 2, 3, 4, 5, 6\}$, yet*
- *if you roll both A and B and sum the result, the probability of getting any number in the set $\{2, 3, \ldots 12\}$ is the **exact same** as if you had rolled two standard dice and summed their results?*

2. A Curious Substitution: The Development

So: often, when we're integrating things, we often come up across expressions like

$$\int_0^\pi \frac{1}{1+\sin(\theta)} d\theta, \text{ or } \int_{-\pi/4}^{\pi/4} \frac{1}{\cos(\theta)} d\theta,$$

where there's no immediately obvious way to set up the integral. Sometimes, we can be particuarly clever, and notice some algebraic trick: for example, last Friday we used the observation that

$$\begin{aligned}\frac{1}{\cos(\theta)} &= \frac{\cos(\theta)}{\cos^2(\theta)}\\ &= \frac{\cos(\theta)}{1-\sin^2(\theta)}\\ &= \frac{1}{2}\left(\frac{\cos(\theta)}{1-\sin(\theta)} + \frac{\cos(\theta)}{1+\sin(\theta)}\right),\end{aligned}$$

in order to integrate $\sec(\theta)$.

Relying on being clever all the time, however, is not usually a winning strategy: it would be nice if we could come up with some way of methodically studying such integrals above – specifically, of working with integrals that feature a lot of trigonometric identities! Is there a way to do this?

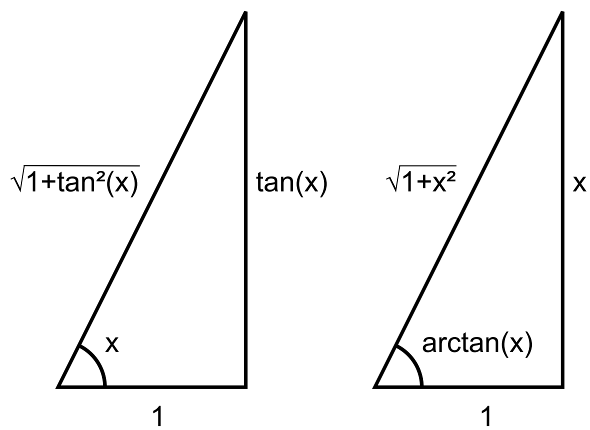
As it turns out: yes! Specifically, consider the use of the following function as a substitution:

$$g(x) = 2\arctan(x),$$

where $\arctan(x)$ is the inverse function to $\tan(x)$, and is a function $\mathbb{R} \to (\pi/2, \pi/2)$. In class, we showed that such inverse functions of differentiable functions are differentiable themselves: consequently, we can use the chain rule and the definition of the inverse to see that

$$\begin{aligned}
&(\tan(\arctan(x))' = (x)' = 1, \text{ and}\\
&(\tan(\arctan(x))' = \tan'(\arctan(x)) \cdot (\arctan(x))' = \frac{1}{\cos^2(\arctan(x))} \cdot (\arctan(x))'\\
\Rightarrow &\frac{1}{\cos^2(\arctan(x))} \cdot (\arctan(x))' = 1\\
\Rightarrow &(\arctan(x))' = \cos^2(\arctan(x)).
\end{aligned}$$

Then, if we remember how the trigonometric functions were defined, we can see that (via the below triangles)



we have that

$$(\arctan(x))' = \cos^2(\arctan(x)) = \frac{1}{1+x^2}$$

and thus that

$$g'(x) = \frac{2}{1+x^2}.$$

As well: by using the above triangles, notice that

$$\begin{aligned}
\sin(g(x)) &= \sin(2\arctan(x))\\
&= 2\sin(\arctan(x))\cos(\arctan(x)\\
&= 2 \cdot \frac{1}{\sqrt{1+x^2}} \cdot \frac{x}{\sqrt{1+x^2}}\\
&= \frac{2x}{1+x^2},
\end{aligned}$$

and

$$\begin{aligned}\cos(g(x)) &= \cos(2\arctan(x)) \\ &= 2\cos^2(\arctan(x)) - 1 \\ &= \frac{2}{1+x^2} - 1 \\ &= \frac{1-x^2}{1+x^2}.\end{aligned}$$

Finally, note that trivially we have that

$$g^{-1}(x) = \tan(x/2),$$

by definition.

What does this all mean? Well: suppose we have some function $f(x)$ where all of its terms are trig functions – i.e. $f(x) = \frac{1}{1+\sin(x)}$, or $f(x) = \frac{1}{\cos(x)}$ – and we make the substiution

$$\int_a^b f(x) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x))g'(x).$$

What do we know about the integral on the right? Well: as we've just shown above, the substitution of $g(x)$ turns all of the $\sin(x)$'s into $\sin(g(x))$'s, which are just reciprocals of polynomials; similarly, we've turned all of the $\cos(x)$'s into $\cos(g(x))$'s, which are also made of polynomials. In other words, this substitution turns a function that's made entirely out of trig functions into one that's made **only out of polynomials!** – i.e. it turns trig functions into quadratic polynomials! This is excellent for us, because (as you may have noticed) it's often *far* easier to integrate polynomials than trig functions.

This substitution is probably one of those things that's perhaps clearer in its use than its explanation. Consequently, we have several examples in the next section to illustrate how this substitution is used:

3. A Curious Substitution: Some Examples of its Use

Example 3.1. Find the integral

$$\int_0^{\pi/2} \frac{1}{1+\sin(x)} dx.$$

Proof. So: without thinking, let's just try our substitution, where $f(x) = \frac{1}{1+\sin(x)}$:

$$\begin{aligned}
\int_0^{\pi/2} \frac{1}{1+\sin(x)} dx &= \int_{g^{-1}(0)}^{g^{-1}(\pi/2)} f(g(x))g'(x)dx \\
&= \int_{\tan(0)}^{\tan(pi/4)} \frac{1}{1+\frac{2x}{1+x^2}} \cdot \frac{2}{1+x^2} dx \\
&= \int_0^1 \frac{2}{1+x^2+2x} dx \\
&= \int_0^1 \frac{2}{(1+x)^2} dx \\
&= \int_1^2 \frac{2}{x^2} dx \\
&= -\frac{2}{x}\bigg|_1^2 \\
&= 1/2.
\end{aligned}$$

...so it works! Without any effort, we were able to just mechanically calculate an integral that otherwise looked nigh-impossible. Neat! □

Example 3.2. Find the integral

$$\int_{-\pi/4}^{\pi/4} \frac{1}{\cos(x)} dx.$$

Proof. Let's try just using our substitution again, where $f(x) = \frac{1}{\cos(x)}$:

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \frac{1}{\cos(x)} dx &= \int_{g^{-1}(-\pi/4)}^{g^{-1}(\pi/4)} f(g(x))g'(x)dx \\
&= \int_{\tan(-\pi/8)}^{\tan(\pi/8)} \frac{1}{\frac{1-x^2}{1+x^2}} \cdot \frac{2}{1+x^2} dx \\
&= \int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{2}{1-x^2} dx \\
&= \int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{2}{(1-x)(1+x)} dx \\
&= \int_{1-\sqrt{2}}^{\sqrt{2}-1} \left(\frac{1}{1-x} + \frac{1}{1+x}\right) dx \\
&= \int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{1}{1-x} dx + \int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{1}{1+x} dx,
\end{aligned}$$

where the trick between the fourth and fifth lines was using partial fractions to simplify the fraction.

Now, use the two u-substitutions $u = 1 - x$ and $u = 1 + x$ on the above two fractions to see that

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \frac{1}{\cos(x)} dx &= \int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{1}{1-x} dx + \int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{1}{1+x} dx \\
&= \int_{\sqrt{2}}^{2-\sqrt{2}} -\frac{1}{u} du + \int_{2-\sqrt{2}}^{\sqrt{2}} \frac{1}{u} du \\
&= \int_{2-\sqrt{2}}^{\sqrt{2}} \frac{1}{u} du + \int_{2-\sqrt{2}}^{\sqrt{2}} \frac{1}{u} du \\
&= 2 \int_{2-\sqrt{2}}^{\sqrt{2}} \frac{1}{u} du \\
&= 2 \ln(u) \Big|_{2-\sqrt{2}}^{\sqrt{2}} \\
&= 2 \ln(\sqrt{2}) - 2 \ln(2 - \sqrt{2}) \\
&= \ln \left(\frac{2}{(2-\sqrt{2})^2} \right) \\
&= \ln \left(\frac{2+\sqrt{2}}{2-\sqrt{2}} \right) \\
&= \ln \left(\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2} \right),
\end{aligned}$$

which agrees exactly with our answer from Friday, and required no clever algebraic tricks to discover! (Well, no clever tricks beyond our useful substitution.) □

However, it bears noting that this substitution is not a miracle worker; there are many functions whose integrals it will not simplify, and indeed some functions which it will make much more complicated. For these reasons, consider it a substitution of "last resort" – if you can't think of anything else to try, go for it, but be aware that it can make some simple integrals far more complex than they need be (as we will see in our last example:)

Example 3.3. Find the integral

$$\int_0^{\pi/2} \sin^2(x) dx.$$

Proof. Suppose that we have forgotten all about the double-angle formula, and just wanted to blindly apply our formula: then, for $f(x) = \sin^2(x)$, we would have

$$\begin{aligned}\int_0^{\pi/2} \sin^2(x)dx &= \int_{g^{-1}(0)}^{g^{-1}(\pi/4)} f(g(x))g'(x)dx \\ &= \int_0^1 \left(\frac{2x}{1+x^2}\right) \cdot \frac{2}{1+x^2}dx \\ &= \int_0^1 \frac{8x^2}{(1+x^2)^3}dx,\end{aligned}$$

which is arguably a much more awful thing to study! As it turns out, we can integrate it via partial fractions:

$$\begin{aligned}\int_0^1 \frac{8x^2}{(1+x^2)^3}dx &= \int_0^1 \left(\frac{8}{(1+x^2)^2} - \frac{8}{(1+x^2)^3}\right)dx \\ &= \int_0^1 \left(\frac{8}{(1+x^2)^2} - \frac{8}{(1+x^2)^3}\right)dx,\end{aligned}$$

which we can calculate via the u-substitution $x = \tan(u), dx = \frac{1}{\cos^2(x)}$:

$$\begin{aligned}&\int_0^1 \left(\frac{8}{(1+x^2)^2} - \frac{8}{(1+x^2)^3}\right)dx \\ =&\int_0^{\pi/4} \left(\frac{8}{(1+\tan^2(u))^2}\cdot\frac{1}{\cos^2(u)} - \frac{8}{(1+\tan^2(u))^3}\cdot\frac{1}{\cos^2(u)}\right)du \\ =&\int_0^{\pi/4} \left(\frac{8}{\cos^{-4}(u)}\cdot\frac{1}{\cos^2(u)} - \frac{8}{\cos^{-6}(u)}\cdot\frac{1}{\cos^2(u)}\right)du \\ =&\int_0^{\pi/4} \left(8\cos^2(u) - 8\cos^4(u)\right)du \\ =&\int_0^{\pi/4} \left(4 + 4\cos(2u) - 2(1+\cos(2u))^2\right)du \\ =&\int_0^{\pi/4} \left(4 + 4\cos(2u)) - 2 - 4\cos(2u) - 2\cos^2(2u)\right)du \\ =&\int_0^{\pi/4} (2 - 1 - \cos(4u))\,du \\ =&\left(u - \frac{\sin(4u)}{4}\right)\Bigg|_0^{\pi/4} \\ =&\pi/4.\end{aligned}$$

Just to check, this does agree completely with the far easier method of just using the double-angle formula:

$$\int_0^{\pi/2} \sin^2(x)dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{2}dx = \frac{x}{2} - \frac{\cos(2x)}{4}\Bigg|_0^{\pi/2} = \pi/4.$$

So: yeah, sometimes this method is a really bad idea. But sometimes (as in the two earlier examples) it's awesome! So don't be afraid to use it, but keep in mind that it's not a panacea; there are many many things that it will not help with. But some things that it will! □