MATH 8, SECTION 1, WEEK 9 - RECITATION NOTES

TA: PADRAIC BARTLETT

Abstract. These are the notes from Monday, Nov. 22nd's lecture, where we started our discussion of Taylor series.

1. Random Question

More gladiatorial arenas! This time: you're in the center of a perfect circle. Somewhere on the boundary of this circle there's a lion, who is restrained by chains to only run along the boundary of the circle. You know the following things to be true:

- The lion runs precisely four times as fast as you can.
- Both you and the lion are point-masses that can change direction instantly.
- Both you and the lion are arbitrarily brilliant.

Can you escape the circle?

2. Taylor Series: Definitions and Theorems

Definition 2.1. Let $f(x)$ be a n-times differentiable function on some neighborhood $(a-\delta, a+\delta)$ of some point a. We define the **n-th Taylor polynomial of $f(x)$ around a** as the following degree-n polynomial:

$$T_n(f(x), a) := \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n.$$

Notice that this function's first n derivatives all agree with $f(x)$'s derivatives: i.e. for any $k \leq n$,

$$\frac{\partial^k}{\partial x^k} \left(T_n(f(x), a)\right) \Big|_a = f^{(k)}(a).$$

This, perhaps, motivates the idea of these Taylor polynomials as "approximations" to the function $f(x)$, and suggests the following definition for the **n-th order remainder function of $f(x)$ around a**:

$$R_n(f(x), a) = f(x) - T_n(f(x), a).$$

If f is an infinitely-differentiable function, and $\lim_{n\to\infty} R_n(f(x), a) = 0$ at some value of x, then we can say that these Taylor polynomials converge to $f(x)$, and in fact write $f(x)$ as its Taylor series at that point:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n.$$

It bears noting that just because a function is infinitely differentiable, its Taylor series can still diverge. (Specifically, in our next lecture, we will discuss a function whose Taylor series only converges at 0, and is completely useless everywhere else.) So be careful!

Theorem 2.2. *(Taylor's theorem:) If $f(x)$ is a $n+1$-times differentiable function on some neighborhood $(a-\delta, a+\delta)$ of some point a and $x > a$ is in $(a-\delta, a+\delta)$, then there is some point c in the neighborhood (a, x) such that*

$$R_n(f(x), a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

The above theorem can be quite useful in determining whether a function's Taylor polynomials will converge to the function itself. For example, we know that

$$R_n(e^x, 0) = \frac{e^c}{(n+1)!}x^{n+1},$$

for some value of $c \in (0, x)$. Consequently, because $n!$ grows much faster than $e^x \cdot x^n$, we know that

$$\forall x \in \mathbb{R}, \lim_{n\to\infty} R_n(e^x, 0) = 0,$$

and thus that

$$e^x = \sum_{n=0}^{\infty} \frac{e^x\Big|_0}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Another example to consider is the function $f(x) = \sin(x)$. By induction, we know that for any n,

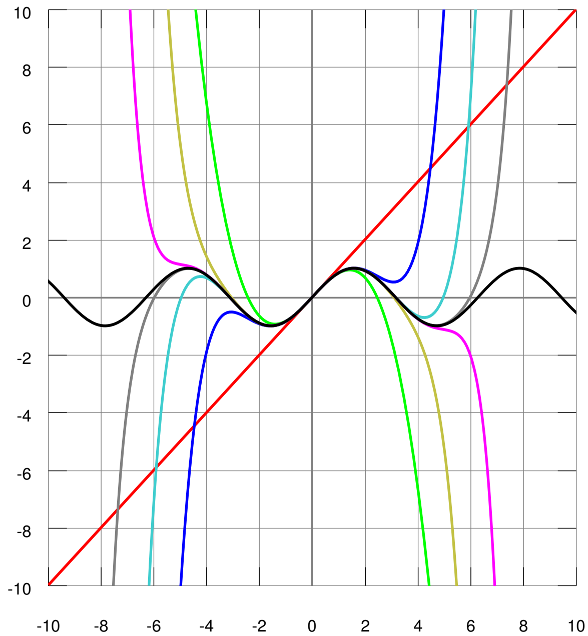
$$\begin{aligned}
\frac{\partial^{4n}}{\partial x^{4n}}(\sin(x))\Big|_0 &= \sin(x)\Big|_0 = 0 \\
\frac{\partial^{4n+1}}{\partial x^{4n+1}}(\sin(x))\Big|_0 &= \cos(x)\Big|_0 = 1 \\
\frac{\partial^{4n+2}}{\partial x^{4n+2}}(\sin(x))\Big|_0 &= -\sin(x)\Big|_0 = 0 \\
\frac{\partial^{4n+3}}{\partial x^{4n+3}}(\sin(x))\Big|_0 &= -\cos(x)\Big|_0 = -1.
\end{aligned}$$

Consequently, we have that the $2n+1$-st Taylor polynomial for $\sin(x)$ around 0 is just

$$T_{2n+1}(\sin(x), 0) = \sum_{k=0}^{n} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

because plugging in the derivatives of $\sin(x)$ into the formula for Taylor polynomials just kills off every other term in the sum.

Graphing several of these polynomials helps describe why these functions can be thought of as kinds of "approximations:"



(picture from Wikipedia)

In the picture above, the polynomials (drawn in color) are increasingly good approximations to $\sin(x)$ near 0. It bears noting that each of these polynomials are still only good approximations *near* 0 – far away from 0, all of these finite-degree polynomials go to $\pm\infty$, while $\sin(x)$ remains bounded by ± 1.

The best way to get a handle on Taylor series is probably to just start using them: in our next section, we do exactly that.

3. Taylor Series: Applications

3.1. **Proving e is Irrational.** This section is devoted to proving the following claim:

Lemma 3.1. *e is irrational.*

Proof. As we often do when we have no idea how to proceed directly: we proceed by contradiction, and assume instead that $e = \frac{a}{b}$, for some pair of positive integers a, b.

Earlier in lecture, we established that

$$T_n(e^x, 0) = \sum_{n=0}^{n} \frac{x^n}{n!};$$

so, if we plug in $x = 1$, we get in fact that

$$\begin{aligned} e^1 = e = \frac{a}{b} &= T_n(e^x, 0)\Big|_1 + R_n(e^x, 0)\Big|_1. \\ &= \left(1 + \frac{x}{1} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right)\Big|_1 + R_n(e^x, 0)\Big|_1. \\ &= \left(1 + \frac{1}{1} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) + R_n(e^x, 0)\Big|_1. \end{aligned}$$

Choose $n > b, 3$, and multiply everything through by $n!$. This yields the equation

$$n!\frac{a}{b} = \left(n! + \frac{n!}{1} + \frac{n!}{2!} + \ldots + \frac{n!}{n!}\right) + n! \cdot \left(R_n(e^x, 0)\Big|_1\right).$$

By our assumption on n, we know that $n!\frac{a}{b}$ is an integer; similarly, we know that all of $\left(n! + \frac{n!}{1} + \ldots + \frac{n!}{n!}\right)$ is also an integer. Consequently, because the sum and difference of integers is always integral, we know that $\left(R_n(e^x, 0)\Big|_1\right)$ is an integer as well!

However, by Taylor's theorem, we know that there is some value of $c \in (0, 1)$ such that

$$R_n(e^x, 0)\Big|_1 = \frac{\frac{\partial^{n+1}}{\partial x^{n+1}}\left(e^x\right)\Big|_c}{(n+1)!} 1^{n+1} = \frac{e^c}{(n+1)!}.$$

We know that $0 < e^c < 3$ for every $c \in (0, 1)$: consequently, we know that

$$\begin{aligned} 0 < R_n(e^x, 0)\Big|_1 &< \frac{3}{(n+1)!} \\ \Rightarrow 0 < n! \cdot R_n(e^x, 0)\Big|_1 &< \frac{3}{n+1} \end{aligned}$$

Because $n > 3$, we know that $\frac{3}{(n+1)} < \frac{3}{4}$: consequently, we in fact have that $n! \cdot R_n(e^x, 0)\Big|_1$ is an integer between 0 and 3/4. As no such thing exists, we have a contradiction! Therefore e must be irrational, as we claimed. □

3.2. **The Binomial Theorem.** In high school, many of you probably saw the following theorem:

Lemma 3.2.

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$.

Here, we will prove a generalization of this theorem:

Lemma 3.3. *For* ***any real number*** $a \in \mathbb{R}$, *define*

$$\binom{a}{k} = \frac{a \cdot (a-1) \cdot \ \ldots \cdot (a-k+1)}{k!}.$$

(In the case where $a \in \mathbb{N}$, *it bears noting that this agrees with our earlier definition for* $\binom{a}{k}$.*)*

Then, for any triple of real numbers x, y, a, *with* $0 < x < y$, *we have*

$$(x + y)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k y^{n-k}.$$

Proof. We first make the following claim:

$$\frac{\partial^k}{\partial x^k}(x+y)^a = a \cdot (a-1) \cdot \ldots \cdot (a-k+1) \cdot (x+y)^{a-k}.$$

We prove this by induction. The base case, $k=1$, is trivial, because

$$\frac{\partial}{\partial x}(x+y)^a = a(x+y)^{a-1}.$$

For the inductive step, assume that

$$\frac{\partial^k}{\partial x^k}(x+y)^a = a \cdot (a-1) \cdot \ldots \cdot (a-k+1) \cdot (x+y)^{a-k},$$

and examine

$$\frac{\partial^{k+1}}{\partial x^{k+1}}(x+y)^a = \frac{\partial}{\partial x}\left(\frac{\partial^k}{\partial x^k}(x+y)^a\right).$$

By the inductive hypothesis, this is just

$$\begin{aligned}&\frac{\partial}{\partial x}\left(a \cdot (a-1) \cdot \ldots \cdot (a-k+1) \cdot (x+y)^{a-k}\right)\\ &= \left(a \cdot (a-1) \cdot \ldots \cdot (a-k+1) \cdot (a-k) \cdot (x+y)^{a-k-1}\right),\end{aligned}$$

which is our inductive claim for $k+1$. So we've proved our claim for all $k \in \mathbb{N}$!

Consequently, if we evaluate at $x=0$, we have that

$$\frac{\partial^k}{\partial x^k}(x+y)^a\Big|_{x=0} = a \cdot (a-1) \cdot \ldots \cdot (a-k+1) \cdot (y)^{a-k},$$

and thus that the n-th Taylor polynomial for $(x+y)^a$, thought of as a function of one variable x, expanded around 0, is

$$\begin{aligned}T_n((x+y)^a, 0) &= \sum_{k=0}^{n} \frac{a \cdot (a-1) \cdot \ldots \cdot (a-k+1) \cdot y^{a-k}}{k!} x^k\\ &= \sum_{k=0}^{n} \frac{a \cdot (a-1) \cdot \ldots \cdot (a-k+1)}{k!} y^{a-k} x^k\\ &= \sum_{k=0}^{n} \binom{a}{k} y^{a-k} x^k.\end{aligned}$$

As well, we know that the n-th remainder polynomial for $(x+y)^a$ is

$$\begin{aligned}R_n((x+y)^a, 0) &= \frac{\frac{\partial^{n+1}}{\partial x^{n+1}}(x+y)^a\Big|_{x=c}}{(n+1)!} x^{n+1}\\ &= \frac{a \cdot (a-1) \cdot \ldots \cdot (a-n)(c+y)^{a-n-1}}{(n+1)!} x^{n+1}\\ &= \binom{a}{n+1} x^{n+1} (c+y)^{a-n-1},\end{aligned}$$

for some $c \in (0, x)$.

Now: notice that for any $k \in \mathbb{N}$, if a is positive, we have that

$$\left|\frac{a-k}{n+1-k}\right| \leq \frac{a}{n+1-k}$$
$$\Rightarrow \left|\binom{a}{n+1}\right| \leq \left(\frac{a}{n+1}\right)^{n+1},$$

and if a is negative, we have

$$\left|\frac{a-k}{n+1-k}\right| \leq \frac{n+1-a}{n+1-k}$$
$$\Rightarrow \left|\binom{a}{n+1}\right| \leq \left(\frac{n+1-a}{n+1}\right)^{n+1} = \left(1-\frac{a}{n+1}\right)^{n+1}.$$

Because

- $\lim_{n\to\infty} \left|\frac{a}{n+1}\right|^{n+1} = 0$ (as $(n+1)^{n+1}$ grows much faster than a^{n+1},) and
- $\lim_{n\to\infty} \left(1-\frac{a}{n+1}\right)^{n+1} = e^{-a}$,

we know that in particular for fixed a, the quantity $\left|\binom{a}{k}\right|$ is bounded above by some constant M_a that doesn't depend on k.

Consequently, we know that

$$R_n((x+y)^a, 0) \leq M_a \cdot x^{n+1}(c+y)^{a-n-1}.$$

Because $x < y$ and $c > 0$, we know that $x < c + y$; consequently, we know that $\lim_{n\to\infty} x^{n+1} \cdot (c+y)^{a-(n+1)}$ must be 0, as the $(c+y)$-part eventually overtakes the x-part. Therefore for very large values of n, these remainder functions $R_n((x+y)^a, 0)$ converge to 0. Consequently, we know that we can write $(x+y)^a$ as just the limit of its Taylor polynomials: i.e. that

$$(x+y)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k y^{n-k}.$$

$\square$