Math 8	Instructor: Padraic Bartlett
Review Session: Final	
Week 10	*Caltech - Fall, 2011*

The Math 1a final is focused on the material we've covered in the second half of the course. Specifically, you will be tested on some subset of the following concepts:

- L'Hôpital's rule, and how to use it to evaluate limits.
- Taking derivatives of functions of the form $(f(x))^{g(x)}$.
- Integration: evaluating definite integrals ($\int_a^b$), improper integrals ($\int_a^\infty$), and indefinite integrals ($\int$) using techniques like integration by parts, integration by substitution, and trig identities.
- Using the integral test to evaluate series for convergence.
- Taylor series: being able to find the Taylor series for a given function, as well as being able to use a Taylor series to approximate the integral of a given function.

We present one problem from each of these areas below. For other examples, feel free to either read the notes from previous weeks, or email me! – I can come up with as many examples as you have time to listen to.

1 Worked Problems

1.1 L'Hôpital's rule / functions of the form $(f(x))^{g(x)}$:

Question 1.1. *Show that the limit*

$$\lim_{x\to 0} \frac{(1-x)^x - 1 + x^2}{x^3}$$

converges to $1/2$.

Proof. We bash this limit repeatedly with L'Hôpital's rule. First, before we can apply L'Hôpital's rule, we must check that its conditions apply. The functions contained in the numerator and denominator are all infinitely differentiable near 0, so this will never be a stumbling block: furthermore, because the numerator and denominator are both continuous/defined at 0, we can evaluate their limits at 0 by just plugging in 0: i.e.

$$\lim_{x\to 0}(1-x)^x - 1 + x^2 = (1-0)^0 - 1 + 0^2 = 1 - 1 = 0, \text{ and}$$
$$\lim_{x\to 0} x^3 = 0^3 = 0.$$

So we've satisfied the conditions for L'Hôpital's rule, and can apply it to our limit:

$$\lim_{x\to 0} \frac{(1-x)^x - 1 + x^2}{x^3} =_{L'H} \lim_{x\to 0} \frac{\frac{d}{dx}\left((1-x)^x - 1 + x^2\right)}{\frac{d}{dx}\left(x^3\right)}.$$

At this point, we recall how to differentiate functions of the form $f(x)^{g(x)}$, where $f(x) > 0$, by using the identity

$$\begin{aligned} (f(x))^{g(x)} &= e^{\ln(f(x))\cdot g(x)} \\ \Rightarrow \frac{d}{dx}(f(x))^{g(x)} &= \frac{d}{dx}e^{\ln(f(x))\cdot g(x)} \\ &= e^{\ln(f(x))\cdot g(x)} \cdot \left(\frac{g(x)}{f(x)} \cdot f'(x) + g'(x)\ln(f(x))\right). \end{aligned}$$

In particular, we can rewrite $(1-x)^x$ as $e^{\ln(1-x)\cdot x}$, which will let us just differentiate using the chain rule:

$$\begin{aligned} \lim_{x\to 0} \frac{(1-x)^x - 1 + x^2}{x^3} &=_{L'H} \lim_{x\to 0} \frac{\frac{d}{dx}\left((1-x)^x - 1 + x^2\right)}{\frac{d}{dx}\left(x^3\right)} \\ &= \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{\ln(1-x)\cdot x} - 1 + x^2\right)}{\frac{d}{dx}\left(x^3\right)} \qquad = \lim_{x\to 0} \frac{e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right) + 2x}{3x^2} \end{aligned}$$

Again, both the numerator and denominator are continuous, and plugging in 0 up top yields $e^{ln(1)\cdot 0} \cdot \left(\ln(1) + \frac{0}{1}\right) - 2\cdot 0 = 0$, while on the bottom we also get 0. Therefore, we can apply L'Hôpital's rule again to get that our limit is just

$$\begin{aligned} &\lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right) + 2x\right)}{\frac{d}{dx}\left(3x^2\right)} \\ &= \lim_{x\to 0} \frac{e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right)^2 + e^{\ln(1-x)\cdot x} \cdot \left(-\frac{1}{1-x} - \frac{1}{(x-1)^2}\right) + 2}{6x} \end{aligned}$$

Again, the top and bottom are continuous near 0, and at 0 the top is

$$e^{\ln(1-0)\cdot 0} \cdot \left(\ln(1-0) + \frac{0}{0-1}\right)^2 + e^{\ln(1-0)\cdot 0} \cdot \left(-\frac{1}{1-0} - \frac{1}{(0-1)^2}\right) + 2 = 0 - 2 + 2 = 0,$$

while the bottom is also 0. So, we can apply L'Hôpital again! This tells us that our limit is in fact

$$\begin{aligned} &\lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right)^2 + e^{\ln(1-x)\cdot x} \cdot \left(-\frac{1}{1-x} - \frac{1}{(x-1)^2}\right) + 2\right)}{\frac{d}{dx}\left(6x\right)} \\ &= \lim_{x\to 0} \frac{\begin{array}{r} e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right)^3 \ +3e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right) \cdot \left(-\frac{1}{1-x} - \frac{1}{(x-1)^2}\right) \\ +e^{\ln(1-x)\cdot x} \cdot \left(\frac{-1}{(x-1)^2} + \frac{2}{(x-1)^3}\right) \end{array}}{6}. \end{aligned}$$

Again, the top and bottom are made out of things that are continuous at 0. Plugging in 0 to the top *this time* gives us -3, while the bottom gives us 6: therefore, the limit is just

$$\frac{-3}{6} = -\frac{1}{2}.$$

So we're done! $\square$

1.2 Taylor series:

Question 1.2. *Approximate the integral*

$$\int_1^2 \frac{\sin(x)}{x} dx.$$

Proof. Recall, from an earlier Math 8 lecture, the Taylor series for $\sin(x)$:

$$T(\sin(x)) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Using this, we can write

$$\frac{\sin(x)}{x} = \frac{T_n(\sin(x))}{x} + \frac{R_n(\sin(x))}{x},$$

and therefore write

$$\int_1^2 \frac{\sin(x)}{x} dx = \int_1^2 \frac{T_n(\sin(x))}{x} dx + \int_1^2 \frac{R_n(\sin(x))}{x} dx.$$

We do this for the same reasons as in our estimation of the Gaussian integral $\int e^{-x^2}$! Specifically, notice that the $\frac{T_n(\sin(x))}{x}$ part is just a polynomial:

$$\frac{T_n(sin(x))}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

which should be easy to integrate.

This leaves the $\frac{R_n(\sin(x))}{x}$ part, which we should be able to bound using Taylor's theorem. Specifically, we have

$$\begin{aligned}
R_n(\sin(x)) &= \int_0^x \frac{\frac{d^n}{dt^n}(\sin(t))}{n!} \cdot (x-t)^n dt \\
\Rightarrow |R_n(\sin(x))| &\leq \int_0^x \frac{\left|\frac{d^n}{dt^n}(\sin(t))\right|}{n!} \cdot |(x-t)^n| \, dt \\
&\leq \int_0^x \frac{1}{n!} \cdot x^n dt \\
&= \left(\frac{x^n}{n!}\right) \cdot t\Big|_0^x \\
&= \frac{x^{n+1}}{n!}.
\end{aligned}$$

Therefore, we can bound the integral $\int_1^2 \frac{R_n(\sin(x))}{x}$ as follows:

$$\left|\int_1^2 \frac{R_n(\sin(x))}{x}dx\right| \leq \int_0^2 \frac{x^n}{n!}dx = \left.\frac{x^{n+1}}{(n+1)!}\right|_1^2 = \frac{2^{n+1}-1}{(n+1)!}.$$

This quantity is $\frac{7}{80} < .1$ at $n = 6$. Therefore, we've proven that

$$\int_1^2 \frac{\sin(x)}{x}dx = \int_1^2 \frac{T_6(\sin(x))}{x}dx,$$

up to $\pm.1$.

So: to find this integral, it suffices to integrate $\frac{T_6(\sin(x))}{x}$. This is trivial:

$$\begin{aligned}
\int_1^2 \frac{T_6(\sin(x))}{x}dx &= \int_1^2 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{x}dx \\
&= \int_1^2 1 - \frac{x^2}{3!} + \frac{x^4}{5!}dx \\
&= \left(x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!}\right)\Big|_1^2 \\
&= \frac{1193}{1800} \\
&\approx .66
\end{aligned}$$

So our integral is about $.66 \pm .1$.

□

1.3 Integral test:

Question 1.3. *Determine which of the two series*

$$\sum_{n=3}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}, \quad \sum_{n=3}^{\infty} \frac{1}{\ln(n)^{\ln(\ln((n))}}$$

converge.

Proof. First, notice that because $\ln(x)$ is an increasing function, so is $\ln(x)^{\ln(x)}$ and $(\ln(x))^{\ln(\ln(x))}$; consequently, the two functions $\frac{1}{\ln(x)^{\ln(x)}}$, $\frac{1}{(\ln(x))^{\ln(\ln(x))}}$ are both decreasing on $[3, \infty)$. As well, because $\ln(x)$ is positive whenever $x > 1$, we know that these functions are both positive on $[3, \infty)$.

Therefore, in either case, we've satisfied the conditions for the integral test, and can use it to determine whether either of these series converge.

We start with the first series, $\sum_{n=3}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$. By the integral test, this series converges if and only if

$$\int_3^{\infty} \frac{1}{(\ln(x))^{\ln(x)}}dx$$

converges.

To evaluate this, perform the u-substitution $u = \ln(x), x = e^u, dx = e^u du$ to see that

$$\begin{aligned} \int_3^\infty \frac{1}{(\ln(x))^{\ln(x)}} dx &= \int_{\ln(3)}^\infty \frac{1}{u^u} \cdot e^u du \\ &= \int_{\ln(3)}^\infty \frac{1}{e^{u \ln(u)}} \cdot e^u du \\ &= \int_{\ln(3)}^\infty \frac{1}{e^{u(\ln(u)-1)}} du. \end{aligned}$$

Now, notice two things:

1. For $u > 9 > e^2$, we have $\ln(u) > 2$, and therefore $\frac{1}{e^{u(\ln(u)-1)}} < \frac{1}{e^{u(2-1)}} = e^{-u}$. This means that
$$\int_9^\infty \frac{1}{e^{u(\ln(u)-1)}} du < \int_9^\infty e^{-u} du = -e^{-u}\Big|_9^\infty = 0 - (-e^{-9}) = \frac{1}{e^9},$$
which in particular is finite.

2. The remaining part,
$$\int_{\ln(3)}^9 \frac{1}{e^{u(\ln(u)-1)}} du$$
is the integral of a continuous function over a finite interval, and is therefore finite and exists.

Therefore, using the linearity of the integral, we know that the whole integral $\int_{\ln(3)}^\infty \frac{1}{u^u} \cdot e^u du$ is finite: therefore, by the integral test, our series

$$\sum_{n=3}^\infty \frac{1}{\ln(n)^{\ln(n)}} converges.$$

We now turn to our other series, $\sum_{n=3}^\infty \frac{1}{\ln(n)^{\ln(\ln((n))}}$. Again, using the integral test, it suffices to decide whether

$$\int_{n=3}^\infty \frac{1}{(\ln(x))^{\ln(\ln(x))}} dx$$

converges. Again: perform the u-substitution $u = \ln(x), x = e^u, dx = e^u du$ to see that

$$\begin{aligned} \int_{n=3}^\infty \frac{1}{(\ln(x))^{\ln(\ln(x))}} dx &= \int_{n=3}^\infty \frac{1}{u^{\ln(u)}} \cdot e^u du \\ &= \int_{n=3}^\infty \frac{1}{e^{(\ln(u))^2}} \cdot e^u du \\ &= \int_{n=3}^\infty e^{u-(\ln(u))^2} du. \end{aligned}$$

Finally, because

$$\lim_{u\to\infty} u - (\ln(u))^2 = \infty,$$

we know that $e^{u-(\ln(u))^2}$ increases without bound, and thus its integral cannot converge. So the series

$$\int_{n=3}^{\infty} \frac{1}{(\ln(x))^{\ln(\ln(x))}} dx$$

diverges by the integral test. $\square$

1.4 Integration methods:

Question 1.4. *Calculate the following two integrals:*

$$\int_0^1 \ln(1+x^2)dx, \quad \int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}} dx.$$

Proof. We begin by studying $\int_0^1 \ln(1+x^2)dx$. Because no substitution looks very promising (as the $1+x^2$ term messes things up,) we are motivated to try integration by parts. In particular, we can remember the trick we used when integrating $\ln(x)$, and set

$$\begin{array}{ll} u = \ln(1+x^2) & dv = dx \\ du = \frac{2x}{1+x^2} & v = x, \end{array}$$

which gives us

$$\int_0^1 \ln(1+x^2)dx = \ln(1+x^2)\cdot x\Big|_0^1 - \int_0^1 \frac{2x^2}{1+x^2} dx$$

A bit of algebra allows us to notice that

$$\begin{aligned} \ln(1+x^2)\cdot x\Big|_0^1 - 2\int_0^1 \frac{x^2}{1+x^2} dx &= \ln(1+x^2)\cdot x\Big|_0^1 - 2\left(\int_0^1 1 - \frac{1}{1+x^2} dx\right) \\ &= \ln(1+x^2)\cdot x\Big|_0^1 - 2x\Big|_0^1 + 2\int_0^1 \frac{1}{1+x^2} dx. \end{aligned}$$

Now, we remember our inverse trig identities, and specifically remember that $\int \frac{1}{1+x^2} dx = \arctan(x)$; combining, we have

$$\begin{aligned} \int_0^1 \ln(1+x^2)dx &= \ln(1+x^2)\cdot x\Big|_0^1 - 2x\Big|_0^1 + 2\arctan(x)\Big|_0^1 \\ &= \ln(2) - 2 + \frac{\pi}{2}. \end{aligned}$$

We now look at $\int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}dx$. Before we can do anything, we have to do some algebra to clean up this function. Specifically, to simplify this expression, we multiply top and bottom by $\sqrt{x+1}-\sqrt{x-1}$, a common algebraic technique used on square-root-involving expressions to clean things up:

$$\begin{aligned}
\int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}dx &= \int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}} \cdot \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}dx \\
&= \int_2^3 \frac{\sqrt{x+1}-\sqrt{x-1}}{(\sqrt{x+1})^2-(\sqrt{x-1})^2}dx \\
&= \int_2^3 \frac{\sqrt{x+1}-\sqrt{x-1}}{x+1-x+1}dx \\
&= \frac{1}{2}\int_2^3 \sqrt{x+1}-\sqrt{x-1}dx \\
&= \frac{1}{2}\int_2^3 \sqrt{x+1}dx - \frac{1}{2}\int_2^3 \sqrt{x-1}dx.
\end{aligned}$$

We now perform a pair of translation-substitutions, setting u = x+1 in the first integral and u = x-1 in the second integral:

$$\begin{aligned}
&= \frac{1}{2}\int_3^4 \sqrt{u}du - \frac{1}{2}\int_1^2 \sqrt{u}du. \\
&= \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right)\Big|_3^4 - \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right)\Big|_1^2 \\
&= \frac{\sqrt{64}-\sqrt{27}-\sqrt{8}+1}{3}.
\end{aligned}$$

$\square$