Math 8	Instructor: Padraic Bartlett
Lecture 4: Power Series; Continuity	
Week 4	*Caltech - Fall, 2011*

1 Random Questions

Question 1.1. *Last week, we showed that the harmonic series*

$$\sum_{n \in \mathbb{N}} \frac{1}{n}$$

diverges.

Show that the sum

$$\sum_{\substack{n \in \mathbb{N}: \\ n \text{ has no } 9 \\ \text{in its digits}}} \frac{1}{n}$$

converges, and specifically converges to something < 80.

Question 1.2. *For any k, define the following sequence of numbers (often called "hailstone" numbers:)*

$$a_0 = k,$$
$$a_{n+1} = \begin{cases} 3a_n + 1, & a_n \text{ odd.} \\ \frac{a_n}{2}, & a_n \text{ even.} \end{cases}$$

Show that for any k, the number 1 eventually shows up in the sequence $\{a_n\}_{n=1}^{\infty}$.

Question 1.3. *Can you find a function $f : \mathbb{R} \to \mathbb{R}$ such that f is*

- *continuous nowhere?*
- *continuous at every point of $\mathbb{Q}$, but not at any point of $\mathbb{R} \setminus \mathbb{Q}$?*
- *continuous at every point of $\mathbb{R} \setminus \mathbb{Q}$, but not at any point of $\mathbb{Q}$?*
- *not continuous at 0, but somehow is linear*[1] *?*

This week's talks focus on two distinct topics that we'll deal with repeatedly in Math 1: the study of **power series**, a "combination" of series and polynomials that have a number of useful properties, and the concepts of **limits** and **continuity** for real-valued functions. We start with power series:

[1]A function $f : \mathbb{R} \to \mathbb{R}$ is linear if $f(x + y) = f(x) + f(y)$, for every $x, y \in \mathbb{R}$.

2 Power Series

2.1 Power Series: Definitions and Tools

The motivation for **power series**, roughly speaking, is the observation that polynomials are really *quite nice*. Specifically, if I give you a polynomial, you can

- differentiate and take integrals easily,
- add and multiply polynomials together and easily express the result as another polynomial,
- find its roots,

and do most anything else that you'd ever want to do to a function! One of the only downsides to polynomials, in fact, is that there are functions that **aren't** polynomials! In specific, the very useful functions

$$\sin(x), \cos(x), \ln(x), e^x, \frac{1}{x}$$

are all not polynomials, and yet are remarkably useful/frequently occuring objects.

So: it would be nice if we could have some way of "generalizing" the idea of polynomials, so that we could describe functions like the above in some sort of polynomial-ish way – possibly, say, as polynomials of "infinite degree?" How can we do that?

The answer, as you may have guessed, is via **power series**:

Definition 2.1. A **power series** $P(x)$ centered at x_0 is just a sequence $\{a_n\}_{n=1}^{\infty}$ written in the following form:

$$P(x) = \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n.$$

Power series are almost taken around $x_0 = 0$: if x_0 is not mentioned, feel free to assume that it is 0.

The definition above says that a power series is just a fancy way of writing down a sequence. This looks like it contradicts our original idea for power series, which was that we would generalize polynomials: in other words, if I give you a power series, you quite certainly want to be able to **plug numbers into it**!

The only issue with this is that sometimes, well . . . you can't:

Example. Consider the power series

$$P(x) = \sum_{n=0}^{\infty} x^n.$$

There are values of x which, when plugged into our power series $P(x)$, yield a series that fails to converge.

Proof. There are many such values of x. One example is $x = 1$, as this yields the series

$$P(x) = \sum_{n=0}^{\infty} 1,$$

which clearly fails to converge; another example is $x = -1$, which yields the series

$$P(x) = \sum_{n=0}^{\infty} (-1)^n.$$

The partial sums of this series form the sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$, which clearly fails to converge[2]. □

So: if we want to work with power series as polynomials, and not just as fancy sequences, we need to find a way to talk about where they "make sense:" in other words, we need to come up with an idea of **convergence** for power series! We do this here:

Definition 2.2. A power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to **converge** at some value $b \in \mathbb{R}$ if and only if the series

$$\sum_{n=0}^{\infty} a_n (b - x_0)^n$$

converges. If it does, we denote this value as $P(b)$.

The following theorem, proven in lecture, is remarkably useful in telling us where power series converge:

Theorem 1. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is a power series that converges at some value $b + x_0 \in \mathbb{R}$. Then $P(x)$ actually converges on every value in the interval $(-b + x_0, b + x_0)$.

In particular, this tells us the following:

Corollary 2. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series centered at 0, and A is the set of all real numbers on which $P(x)$ converges. Then there are only three cases for A: either

[2] Though it **wants** to converge to 1/2. Go to wikipedia and read up on Grandi's series for more information!

1. $A = \{0\}$,

2. $A =$ one of the four intervals $(-b, b), [-b, b), (-b, b], [-b, b]$, for some $b \in \mathbb{R}$, or

3. $A = \mathbb{R}$.

We say that a power series $P(x)$ has **radius of convergence** 0 in the first case, b in the second case, and ∞ in the third case.

A question we could ask, given the above corollary, is the following: can we actually get all of those cases to occur? I.e. can we find power series that converge only at 0? On all of $\mathbb{R}$? On only an open interval?

To answer these questions, consider the following examples:

2.2 Power Series: Examples

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} n! \cdot x^n$$

converges when $x = 0$, and diverges everywhere else.

Proof. That this series converges for $x = 0$ is trivial, as it's just the all-0 series.

To prove that it diverges whenever $x \neq 0$: pick any $x > 0$. Then the ratio test says that this series diverges if the limit

$$\lim_{n\to\infty} \frac{(n+1)!x^{n+1}}{n! \cdot x^n} = \lim_{n\to\infty} x(n+1) = +\infty$$

is > 1, which it is. So this series diverges for all $x > 0$. By applying our theorem about radii of convergence of power series, we know that our series can only converge at 0: this is because if it were to converge at any negative value $-x$, it would have to converge on all of $(-x, x)$, which is a set containing positive real numbers. $\square$

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} x^n$$

converges when $x \in (-1, 1)$, and diverges everywhere else.

Proof. Take any $x > 0$, as before, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}}{x^n} = x.$$

So the series diverges for $x > 1$ and converges for $0 \leq x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$: in our earlier discussion of power series, we proved that $P(x)$ diverged at both 1 and -1. So this power series converges on $(-1, 1)$ and diverges everywhere else. $\square$

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges when $x \in [-1, 1)$, and diverges everywhere else.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)}{x^n/n} = \lim_{n\to\infty} x \cdot \frac{n}{n+1} = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right) = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \le x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the harmonic series (which diverges) and plugging in -1 yields the alternating harmonic series (which converges.) So this power series converges on $[-1, 1)$ and diverges everywhere else. □

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

converges when $x \in [-1, 1]$, and diverges everywhere else.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)^2}{x^n/n^2} = \lim_{n\to\infty} x \cdot \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right)^2 = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \le x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the series $\sum \frac{1}{n^2}$, which we've shown converges. Plugging in -1 yields the series $\sum \frac{(-1)^n}{n^2}$: because the series of termwise-absolute-values converges, we know that this series converges absolutely, and therefore converges.

So this power series converges on $[-1, 1]$ and diverges everywhere else. □

Example. The power series

$$P(x) = \sum_{n=0}^{\infty} 0 \cdot x^n$$

converges on all of $\mathbb{R}$.

Proof. $P(x) = 0$, for any x, which is an *exceptionally* convergent series. □

Example. The power series

$$P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges on all of $\mathbb{R}$.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

So this series converges for any $x > 0$: applying our theorem about radii of convergence tells us that this series must converge on all of $\mathbb{R}$! $\square$

This last series is particularly interesting, as you'll see later in Math 1. One particularly nice property it has is that $P(1) = e$:

Definition 2.3.

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Using this, we can prove something we've believed for quite a while but never yet demonstrated:

Theorem 2.4. *e is irrational.*

Proof. We begin with a (somewhat dumb-looking) lemma:

Lemma 3. $e < 3$.

Proof. To see that $e < 3$, look at $e - 2$, factor out a $\frac{1}{2}$, and notice a few basic inequalities:

$$\begin{aligned} e-1-1 =& \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right) - 1 - 1 \\ =& \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \\ =& \frac{1}{2} \cdot \left(1 + \frac{1}{3} + \frac{1}{3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \dots\right) \\ <& \frac{1}{2} \cdot \left(1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \dots\right) \\ =& \frac{1}{2} \cdot \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right) \\ =& \frac{1}{2} \cdot (e-1) \\ \Rightarrow \quad 2e - 4 <& e - 1 \\ \Rightarrow \quad & e < 3. \end{aligned}$$

$\square$

Given this, our proof is remarkably easy! Assume that $e = \frac{a}{b}$, for some pair of integers $a, b \in \mathbb{Z}, b \geq 1$. Then we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} &= \frac{a}{b} \\ \Rightarrow \quad \sum_{n=0}^{\infty} \frac{b!}{n!} &= a \cdot (b-1)! \\ \Rightarrow \quad \sum_{n=0}^{b} \frac{b!}{n!} + \sum_{n=b+1}^{\infty} \frac{b!}{n!} &= a \cdot (b-1)! \\ \Rightarrow \quad \sum_{n=b+1}^{\infty} \frac{b!}{n!} &= a \cdot (b-1)! - \sum_{n=0}^{b} \frac{b!}{n!}. \end{aligned}$$

For $n \leq b$, notice that $\frac{b!}{n!}$ is always an integer: therefore, the right-hand-side of the last equation above is always an integer, as it's just the difference of a bunch of integers. This means, in particular, that the left-hand-side $\sum_{n=b+1}^{\infty} \frac{b!}{n!}$ is *also* an integer. What integer is it?

Well: we know that

$$0 < \frac{1}{b} < \sum_{n=b+1}^{\infty} \frac{b!}{n!} = \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} \ldots,$$

so it's a positive integer.

However, we also know that because $b \geq 1$, we have

$$\begin{aligned} \sum_{n=b+1}^{\infty} \frac{b!}{n!} &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} \ldots \\ &\leq \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots \\ &= \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \\ &= e - 2 \qquad < 1. \end{aligned}$$

So, it's an integer strictly between 0 and …1. As there are no integers strictly between 0 and 1, this is a contradiction! – in other words, we've just proven that e must be rational. □

3 Continuity

Changing gears here, we turn to the concepts of **continuity** and **limits** of real-valued functions:

3.1 Continuity: Motivation and Tools

Definition 3.1. If $f : X \to Y$ is a function between two subsets X, Y of $\mathbb{R}$, we say that

$$\lim_{x \to a} f(x) = L$$

if and only if

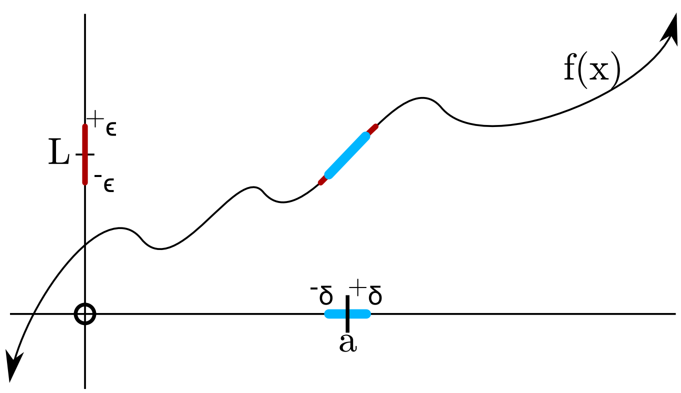
1. (vague:) as x approaches a, $f(x)$ approaches L.
2. (precise; wordy:) for any distance $\epsilon > 0$, there is some bound $\delta > 0$ such that whenever $x \in X$ is within δ of a, $f(x)$ is within ϵ of L.
3. (precise; symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in X, (|x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

Definition 3.2. A function $f : X \to Y$ is said to be **continuous** at some point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a).$$

These definitions, without pictures, are kind-of hard to understand. In high school, continuous functions are often simply described as "functions you can draw without lifting your pencil[3];" how do these deltas and epsilons relate to this intuitive concept? Consider the following picture:



This graph should help to illustrate what's going on in our "rigorous" definition of limits and continuity. Essentially, when we claim that "as x approaches a, $f(x)$ approaches $f(a)$", we are saying

- for any (red) distance ϵ around $f(a)$ that we'd like to keep our function,
- there is a (blue) neighborhood $(a - \delta, a + \delta)$ around a such that
- if f takes only values within this (blue) neighborhood $(a - \delta, a + \delta)$, it stays within the (red) ϵ neighborhood of $f(a)$.

[3] Assuming, of course, an arbitrarily sharp pencil, infinite amounts of lead, and a sheet of paper the size of $\mathbb{R}^2$ to draw on.

Basically, what this definition says is that if you pick values of x sufficiently close to a, the resulting $f(x)$'s will be as close as you want to be to $f(a)$ – i.e. that "as x approaches a, $f(x)$ approaches $f(a)$."

This, hopefully, illustrates what our definition is trying to capture – a concrete notion of something like convergence for functions, instead of sequences. So: how can we prove that a function f has some given limit L? Motivated by this analogy to sequences, we have the following blueprint for a proof-from-the-definition that $\lim_{x\to a} f(x) = L$:

1. First, examine the quantity

$$|f(x) - L|.$$

   Specifically, try to find a simple upper bound for this quantity that depends only on $|x-a|$, and goes to 0 as x goes to a – something like $|x-a| \cdot$ (constants), or $|x-a|^3 \cdot$ (bounded functions, like $\sin(x)$).

2. Using this simple upper bound, for any $\epsilon > 0$, choose a value of δ such that whenever $|x-a| < \delta$, your simple upper bound $|x-a| \cdot$ (constants) is $< \epsilon$. Often, you'll define δ to be ϵ/(constants), or somesuch thing.

3. Plug in the definition of the limit: for any $\epsilon > 0$, we've found a δ such that whenever $|x-a| < \delta$, we have

$$|f(x) - L| < (\text{simple upper bound depending on } |x-a|) < \epsilon.$$

   Thus, we've proven that $\lim_{x\to a} f(x) = L$, as claimed.

Limits and continuity are wonderfully useful concepts, but working with them straight from the definitions can be somewhat ponderous. As a result, just like we did for sequences, we have developed a number of useful tools and theorems to allow us to prove that certain limits exist without going through the definition every time. We present four such tools here:

1. **Squeeze theorem**: Suppose that f, g, h are functions defined on some interval $I \setminus \{a\}$[4] such that

$$f(x) \le g(x) \le h(x), \forall x \in I \setminus \{a\},$$
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x).$$

   Then $\lim_{x\to a} g(x)$ exists, and is equal to the other two limits $\lim_{x\to a} f(x), \lim_{x\to a} h(x)$.

[4] The set $X \setminus Y$ is simply the set formed by taking all of the elements in X that are not elements in Y. The symbol $\setminus$, in this context, is called "set-minus", and denotes the idea of "taking away" one set from another.

2. **Limits and arithmetic**: Suppose that f, g are a pair of functions such that the limits $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist. Then we have the following equalities:

$$\lim_{x\to a}(f(x)+g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right).$$
$$\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$
$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \left(\lim_{x\to a} f(x)\right) / \left(\lim_{x\to a} g(x)\right), \text{ if } \lim_{x\to a} g(x) \neq 0.$$

As a special case, the product and sum of any two continuous functions is continuous, as is dividing a continuous function by another continuous function that's never zero.

3. **Limits and composition**: Suppose that $f : Y \to Z$ is a function such that $\lim_{y\to a} f(x) = L$, and $g : X \to Y$ is a function such that $\lim_{x\to b} g(x) = a$. Then

$$\lim_{x\to b} f(g(x)) = L.$$

Specifically, if two functions are continuous, their composition is continuous.

4. **Discontinuous functions and sequences**: For any function $f : X \to Y$, we know that f is discontinuous at a point $a \in \mathbb{R}$ if and only if there is some sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

   - $\lim_{n\to\infty} a_n = a$, and
   - $\lim_{n\to\infty} f(a_n) \neq f(a)$.

We illustrate how to use the definition of continuity, as well as how to use each of these four tools, in the next section:

3.2 Continuity: Examples

Claim 4. The function $f(x) = x^2$ is continuous at $x = 1$.

Proof. (Using the definition of continuity): We want to prove that $\lim_{x\to 1} x^2 = 1^2 = 1$. To do this, we'll try using our blueprint for $\epsilon - \delta$ proofs:

1. First, let's examine the quantity $|f(x) - f(1)| = \left|x^2 - 1\right|$. As stated in the blueprint, our first goal is to bound this above by something simple, multipled by $|x - 1|$. We proceed by blindly trying whichever algebraic tricks come to mind:

$$\begin{aligned} |x^2 - 1| &= |(x-1)(x+1)| \\ &= |x-1| \cdot |x+1| \end{aligned}$$
.

   By algebraic simplification, we've broken our expression into two parts: one of which is $|x - 1|$, and the other of which is. . . something. We'd like to get rid of this extra part $|x+1|$; so, how do we do this? We cannot just say that this quantity is bounded; indeed, for very large values of x, this explodes off to infinity.

However, for values of x rather close to 1, this is bounded! In fact, if we have values of x such that x is distance ≤ 1 from the real number 1, we have that $|x+1| \leq 2$.

So, when we pick our δ, if we just make sure that $\delta < 1$, we know that we have the following simple and excellent upper bound:

$$|f(x) - f(a)| \leq 2|x - a|$$

2. We have a simple upper bound! Our next step then proceeds as follows: for any $\epsilon > 0$, we want to pick a $\delta > 0$ such that if $|x - a| < \delta$,

$$|x - a| \cdot 2 < \epsilon.$$

But this is easy: if we want this to happen, we just need to pick δ so that $\delta < 1$ (so we get our simple upper bound,) and also so that $\delta < \frac{\epsilon}{2}$. Explicitly, we can pick $\delta < \min\left(1, \frac{\epsilon}{2}\right)$.

3. Thus, for any $\epsilon > 0$, we've found a $\delta > 0$ such that whenever $|x - 1| < \delta$, we have

$$|f(x) - f(1) \leq 2|x - 1| < \epsilon.$$

Therefore $f(x) = x^2$ is continuous at 1, as claimed.

□

Claim 5. Every polynomial is continuous everywhere.

Proof. (Using arithmetic and continuity:) First, notice the following lemma:

Lemma 6. The function $f(x) = x$ is continuous everywhere.

Proof. To prove this, we simply need to show the following:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in X, (|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \epsilon).$$

But we know that $f(x) = x$ and $f(a) = a$: so we're really trying to prove

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in X, (|x - a| < \delta) \Rightarrow (|x - a| < \epsilon).$$

So. Um. Just pick $\delta = \epsilon$. Then, whenever $|x - a| < \delta$, we definitely have $|x - a| < \epsilon$, because **delta and epsilon are the same.** □

Similarly, and with even less effort, you can show that any constant function $g_c(x) = c$ is also continuous: for any $\epsilon > 0$, let δ be **anything**, and your $\epsilon - \delta$ statement will hold!

Believe it or not, the rest of the proof is even more trivial. We have that the functions $f(x) = x$ and $g_c(x) = c$ are both continuous. By multiplying and adding these functions together, we can create any polynomial; thus, by using our theorems on arithmetic and limits, we have shown that any polynomial must be continuous. □

In very specific, we have that x^2 is continuous at 1, which provides a much shorter proof of our earlier result. This hopefully illustrates something very relevant about continuity: if you can use a theorem instead of working from the definition, do so! It will make your life much easier.

Claim 7.

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$

Proof. (Using the squeeze theorem:) So: for all $x \in \mathbb{R}, x \neq 0$, we have that

$$\begin{aligned} & -1 \leq \sin(1/x) \leq 1 \\ \Rightarrow & -x^2 \leq x^2 \sin(1/x) \leq x^2. \end{aligned}$$

By the squeeze theorem, because the limit as $x \to 0$ of both $-x^2$ and x^2 is 0, we have that

$$\lim_{x \to 0} x^2 \sin(1/x) = 0$$

as well. □

Claim 8. $x^2 \sin(1/x^2)$ is continuous on $\mathbb{R} \setminus \{0\}$.

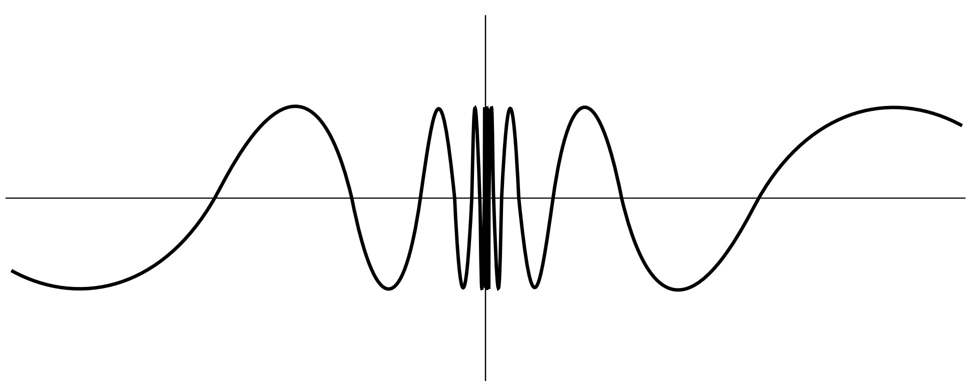
Proof. (Using composition of limits:) By our work earlier in this lecture, x^2 is continuous, and therefore $1/x^2$ is continuous on all of $\mathbb{R} \setminus \{0\}$, by using arithmetic and limits. From Apostol, we know that $\sin(x)$ is continuous: therefore, we have that the composition of these functions $\sin(1/x^2)$ is continuous on $\mathbb{R} \setminus \{0\}$. Multiplying by the continuous function x^2 tells us that $x^2 \sin(1/x^2)$ is continuous on $\mathbb{R} \setminus \{0\}$, as claimed. □

Claim 9. Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ a, & x = 0 \end{cases}$$

Then, no matter what a is, $f(x)$ is discontinuous at 0.

Proof. (Using sequences to show discontinuity:) Before we start, consider the graph of $\sin(1/x)$:



Visual inspection of this graph makes it clear that $\sin(1/x)$ cannot have a limit as x approaches 0; but let's rigorously prove this using our lemma, so we have an idea of how to do this in general.

So: we know that $\sin\left(\frac{4k+1}{2}\pi\right) = 1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+1)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+1)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+1)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+1}{2}\pi\right) = \lim_{k\to\infty} 1 = 1$,

our tool says that if $\sin(1/x)$ has a limit at 0, it must be 1.

However: we also know that $\sin\left(\frac{4k+3}{2}\pi\right) = -1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+3)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+3)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+3)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+3}{2}\pi\right) = \lim_{k\to\infty} -1 = -1$,

our tool **also** says that if $\sin(1/x)$ has a limit at 0, it must be -1. Thus, because $-1 \neq 1$, we have that the limit $\lim_{x\to 0} \sin(1/x)$ cannot exist, as claimed. $\square$