Math 8 — Instructor: Padraic Bartlett

Lecture 5:Discontinuity, the IVT; Differentiability

Week 5 — *Caltech - Fall, 2011*

1 Random Questions

(coming soon!)

This week's notes are currently only partially up; the IVT and differentiability talks will go up hopefully later today. For your midterm, however, the only section of these notes you'll need is below: a discussion of a remarkably discontinuous function!

2 A Remarkably Discontinuous Function

Today, we will consider the following functions:

$$f(x) = \begin{cases} 0, & x \in \mathbb{Z}, \\ 1, & x \notin \mathbb{Z}. \end{cases},$$
$$g(x) = \prod_{n=0}^{\infty} f\left(x \cdot 2^n\right).$$

Specifically, consider the function $g(x)$. Where is it continuous? Where is it discontinuous?

To answer these questions, it might first be useful to even know what our function *looks like*. What are some of its values?

Well: let's plug in some values!

$$g(0) = \prod_{n=0}^{\infty} f\left(0 \cdot 2^n\right) = \prod_{n=1}^{\infty} f(0) = 0 \cdot 0 \cdot \ldots = 0,$$
$$g(1) = \prod_{n=0}^{\infty} f\left(1 \cdot 2^n\right) = f(1) \cdot f(2) \cdot f(4) \cdot \ldots = 0 \cdot 0 \cdot \ldots = 0,$$
$$g(z) = \prod_{n=0}^{\infty} f\left(z \cdot 2^n\right) = f(z) \cdot f(2z) \cdot f(4z) \cdot \ldots = 0 \cdot 0 \cdot \ldots = 0, \forall z \in \mathbb{Z}.$$

So: our function is identically 0 on the integers. What about some other values? Well: if we plug in $\frac{1}{2}$, we get

$$g\left(\frac{1}{2}\right) = \prod_{n=0}^{\infty} f\left(\frac{1}{2}\cdot 2^n\right) = f\left(\frac{1}{2}\right)\cdot f(1)\cdot f(2)\ldots = 1\cdot 0\cdot 0\ldots = 0.$$

So it's zero on $\frac{1}{2}$ as well. In fact, we can easily generalize this to notice that $g(x)$ is zero whenever x is of the form $\frac{a}{2^n}$, for $a \in \mathbb{Z}$, $n \in \mathbb{N}$, as the n-th term $f(x \cdot 2^n)$ will always be zero in our infinite product.

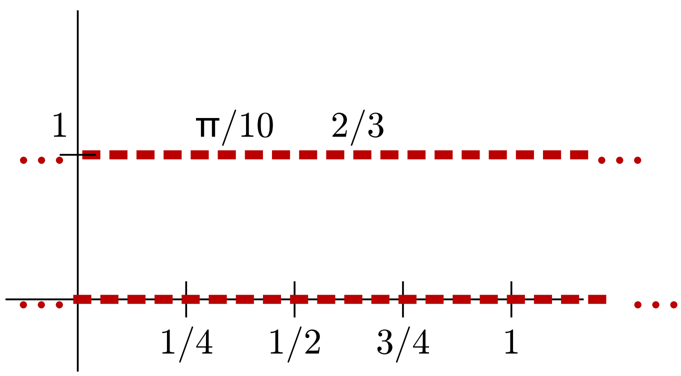
Are there values where $g(x)$ is nonzero? Certainly: take, for example, $x = \frac{1}{3}$:

$$g\left(\frac{1}{3}\right) = \prod_{n=0}^{\infty} f\left(\frac{1}{3}\cdot 2^n\right) = f\left(\frac{1}{3}\right)\cdot f\left(\frac{2}{3}\right)\cdot f\left(\frac{4}{3}\right)\ldots = 1\cdot 1\cdot 1\ldots = 1.$$

In fact, we can see that whenever x is *not* of the form $\frac{a}{2^n}$, we have $g(x) = 1$: i.e. we've shown

$$g(x) = \begin{cases} 0, & x = \frac{a}{2^n} \text{ for some } a \in \mathbb{Z}, n \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases},$$

What does this function look like? Well: let's graph it!



Visually, this function certainly looks like it cannot be continuous **anywhere**: there seem to be values where $g(x)$ is 0 or 1 near every real number! How can we prove this – i.e. how can we prove that our function is nowhere continuous?

Well: recall the lemma we proved in Math 1, about how to prove something is discontinuous:

Lemma 1. For any function $f : X \to Y$, we know that f is discontinuous at a point $a \in \mathbb{R}$ if and only if there is some sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $\lim_{n\to\infty} a_n = a$, and
- $\lim_{n\to\infty} f(a_n) \neq f(a)$.

Using this lemma, it suffices to find for every $x \in \mathbb{R}$ a sequence $\{x_n\}$ that converges to x, but such that the limit $\lim_{n\to\infty} g(x_n) \neq g(x)$.

Pick any $x \in \mathbb{R}$. There are two cases:

1. $x = \frac{a}{2^n}$, for some a, n. In this case, we want a sequence of x_n's that will converge to $\frac{a}{2^n}$, but such that the $g(x_n)$'s will **not** converge to $g(x) = 0$.

   How can we make sure that the $g(x_n)$'s don't converge to 0? Well: if we make sure that every x_n is irrational, then in specific none of the x_n's are of the form $\frac{a}{2^n}$, and therefore $g(x_n) = 1$ for every n, which will certainly make $\lim_{n\to\infty} g(x_n) = 1 \neq 0$.

   We have reduced our problem to the following: we want to find a sequence of irrational numbers that converges to $x = \frac{a}{2^n}$. But this is trivial! Just let

   $$x_n = \frac{a}{2^n} + \frac{\pi}{10^n}.$$

   Then $\lim_{n\to\infty} x_n = x$, while $\lim_{n\to\infty} g(x_n) = 1 \neq 0 = g(x)$. Therefore, we've proven that our function is discontinuous at x, whenever $x = \frac{a}{2^n}$.

2. Suppose now that x is **not** of the form $\frac{a}{2^n}$. Using reasoning similar to the above, we seek to find a sequence of points x_n such that

   - $\lim_{n\to\infty} x_n = x$, while
   - every x_n is of the form $\frac{a}{2^m}$, for some a, m, because this will force $\lim_{n\to\infty} g(x_n) = 0 \neq 1 = g(x)$.

   How can we do this? Well: take x, and write it as a binary[1] string:

   $$x = b_{int} \, . \, b_0 b_1 b_2 b_3 \ldots,$$

   where $b_{int} \in \mathbb{Z}$ and the b_i's are all either 0 or 1. Then look at the sequence of binary approximations to x:

   $$\begin{aligned} x_0 &= b_{int} \cdot 2^0 \\ x_1 &= b_{int} \cdot 2^0 + \frac{b_1}{2^1} \\ x_2 &= b_{int} \cdot 2^0 + \frac{b_1}{2^1} + \frac{b_2}{2^2} \\ &\vdots \end{aligned}$$

[1] We normally write numbers in **decimal** notation, where each digit denotes how many of that power of ten we have in our number. For example, when we write 31.415, we mean

$$31.415_{(\text{decimal})} = 3 \cdot 10^1 + 1 \cdot 10^0 + 4 \cdot 10^{-1} + 1 \cdot 10^{-2} + 5 \cdot 10^{-3}.$$

Binary notation is a similar concept, except we use powers of 2 instead of powers of 10: i.e. we would write 9/4 as $10.01_{(\text{binary})}$, because

$$10.01_{(\text{binary})} = 1 \cdot 2^1 + 0 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} = 2 + \frac{1}{4} = 2.25_{(\text{decimal})}.$$

These numbers are all of the form $\frac{a}{2^m}$ for some m by construction: furthermore, each x_n differs from x at (at worst) all of the digits past the n-th. But this error is at most

$$\sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}},$$

which goes to 0 as n goes to infinity. So this sequence converges to x, while $\lim_{n\to\infty} g(x_n) = 0 \neq 1 = g(x)$: i.e. we've proven that g is discontinuous at x again!

So our function is discontinuous everywhere, as claimed.