Lecture 8: Esoteric Integrations

1   Random Questions

Question 1.1. Can you find a sequence $\{a_k\}_{k=1}^n$, made of the symbols $\{0, 1, 2\}$, that is **repetition-free**: i.e. a sequence in which strings like "11", "1212", "12021202" never show up?

Recall the following definitions:

Definition 1.2. A **graph** G with n vertices and m edges consists of the following two objects:

1. a set $V = \{v_1, \ldots v_n\}$, the members of which we call G's **vertices**, and

2. a set $E = \{e_1, \ldots e_m\}$, the members of which we call G's **edges**, where each edge e_i is an unordered pair of distinct elements in V, and no unordered pair is repeated. For a given edge $e = \{v, w\}$, we will often refer to the two vertices v, w contained by e as its endpoints.

A k-**coloring** of the graph G is a way of assigning each element $v \in V$ to a "color" $\{1, 2, \ldots k\}$, so that whenever $\{u, v\}$ is an edge in E, the color of u is distinct from the color of v.

Question 1.3. Consider the following graph:

- $V = \mathbb{R}^2$.

- $E = \{$ pairs of points $(a, b), (c, d)$ such that the distance $d((a, b), (c, d)) = 1\}$.

Show that this graph is not 3-colorable, but that it is 7-colorable. (These bounds, by the way, are the best known: you can prove without too much effort both of these bounds in a day, while modern mathematics has not came up with any better bounds in years of study.)

Question 1.4. (Something that came up this week in my research:) Let A and B be a pair of finite sets, and $<_*$ be some ordering on the set $A \cup B$. (For example, you could have $A = \{\alpha, \beta, \gamma\}$, $B = \{x, y\}$, and $<_*$ be the ordering $\alpha <_* y <_* \gamma <_* x <_* \beta$.)

Now, look at the set A combined with n distinct copies of B: i.e. $A \cup B_1 \cup B_2 \cup \ldots \cup B_n$. Take a random ordering $<_x$ of the elements in this set. What are the odds that it agrees with the ordering $<_*$ on the set A, but disagrees with it on the sets $A \cup B_i$, for every i?

2 Esoteric Integration Tricks

Last week, we discussed several techniques for integration, discussing (in particular) how to use integration by parts and integration by substitution to study several problems. In these lectures, we discuss some more esoteric techniques and tricks for integration: specifically, we talk about trigonometric substitutions, how to undo trigonometric substitutions, the gaussian integral, and polar/parametric integrals.

2.1 Trigonometric Substitutions

Usually, when people use integration by substitution, they use it to **take functions out** rather than to put functions in. I.e. people usually start with integrals of the form

$$\int_a^b f(g(x))g'(x)dx,$$

and turn them into integrals of the form

$$\int_{g(a)}^{g(b)} f(x)dx.$$

However: this is not the only way to use integration by substitution! Specifically, it is possible to use integration by substitution to **put a g(x) into an integral**, as well! In other words, if we have an integral of the form

$$\int_a^b f(x)dx,$$

we can use integration by substitution to turn it into an integral of the form

$$\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x))g'(x)dx,$$

as long as we make sure that g is continuous on this new interval $[g^{-1}(a), g^{-1}(b)]$.

Why would you want to do this? Well: suppose you're working with a function of the form

$$\frac{1}{a^2 + x^2}.$$

Substituting $x = a\tan(\theta)$ then turns this expression into

$$\frac{1}{a^2 + a^2\tan^2(\theta)} = \frac{1}{a^2\left(1 + \frac{sin^2(\theta)}{\cos^2(\theta)}\right)} = \frac{\cos^2(theta)}{a^2\left(\cos^2(\theta) + \sin^2(\theta)\right)} = \frac{1}{a^2}\cos^2(\theta),$$

which is much simpler. As well: if you have a function of the form

$$\sqrt{a^2 - x^2},$$

the substitution $x = a\sin(\theta)$ turns this into

$$\sqrt{a^2 - a^2 \sin^2(\theta)} = |a| \cdot \sqrt{1 - \sin^2(\theta)} = |a| \cdot \sqrt{\cos^2(\theta)} = |a\cos(\theta)|,$$

which is again a simpler and easier thing to work with! These substitutions come up frequently enough that we refer to them as the **trigonometric substitutions**; they're pretty useful for dealing with many kinds of integrals.

We illustrate their use in the following example:

Question 2.1. What is

$$\int_0^1 \left(x^2 + 1\right)^{-3/2} \quad ?$$

Proof. Looking at this, we see that we have a $\frac{1}{1+x^2}$ term, surrounded by some other bits and pieces. So: let's try the tangent substitution we talked about earlier! Specifically: let

$$f(x) = \left(x^2 + 1\right)^{-3/2}, \quad g(x) = \tan(x),$$
$$g'(x) = \frac{1}{\cos^2(x)}.$$

Then, we have that

$$\int_0^1 \left(x^2 + 1\right)^{-3/2} dx = \int_0^1 f(x)dx$$
$$= \int_{g^{-1}(0)}^{g^{-1}(1)} f(g(x))g'(x)dx$$
$$= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \cos^3(x) \cdot \frac{1}{\cos^2(x)} dx$$
$$= \int_0^{\pi/4} \cos(x)dx$$
$$= \sin(x)\Big|_0^{\pi/4} dx$$
$$= \frac{\sqrt{2}}{2}.$$

$\square$

2.2 Undoing Trigonometric Substitutions

So: often, when we're integrating things, we often come up across expressions like

$$\int_0^\pi \frac{1}{1 + \sin(\theta)} d\theta, \text{ or } \int_{-\pi/4}^{\pi/4} \frac{1}{\cos(\theta)} d\theta,$$

3

where there's no immediately obvious way to set up the integral. Sometimes, we can be particuarly clever, and notice some algebraic trick: for example, to integrate $\frac{1}{\cos(\theta)}$, we can use partial fractions to see that

$$
\begin{aligned}
\frac{1}{\cos(\theta)} &= \frac{\cos(\theta)}{\cos^2(\theta)} \\
&= \frac{\cos(\theta)}{1 - \sin^2(\theta)} \\
&= \frac{1}{2} \left(\frac{\cos(\theta)}{1 - \sin(\theta)} + \frac{\cos(\theta)}{1 + \sin(\theta)} \right),
\end{aligned}
$$

and then integrate each of these two fractions separately with the substitutions $u = 1 \pm \sin(\theta)$.

Relying on being clever all the time, however, is not a terribly good strategy. It would be nice if we could come up with some way of methodically studying such integrals above – specifically, of working with integrals that feature a lot of trigonometric identities! Is there a way to do this?

As it turns out: yes! Specifically, consider the use of the following function as a substitution:

$$
g(x) = 2\arctan(x),
$$

where $\arctan(x)$ is the inverse function to $\tan(x)$, and is a function $\mathbb{R} \to (-\pi/2, \pi/2)$. In class, we showed that such inverse functions of differentiable functions are differentiable themselves: consequently, we can use the chain rule and the definition of the inverse to see that

$$
(\tan(\arctan(x))' = (x)' = 1, \text{ and}
$$

$$
(\tan(\arctan(x))' = \tan'(\arctan(x)) \cdot (\arctan(x))' = \frac{1}{\cos^2(\arctan(x))} \cdot (\arctan(x))'
$$

$$
\Rightarrow \frac{1}{\cos^2(\arctan(x))} \cdot (\arctan(x))' = 1
$$

$$
\Rightarrow (\arctan(x))' = \cos^2(\arctan(x)).
$$

Then, if we remember how the trigonometric functions were defined, we can see that (via the below triangles)

4
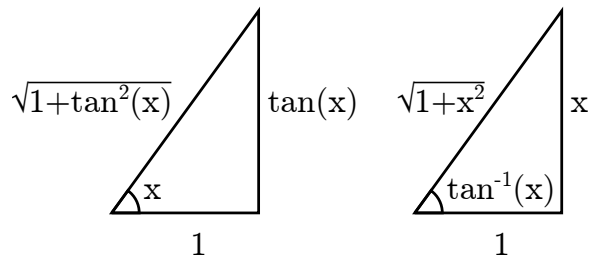
we have that

$$(\arctan(x))' = \cos^2(\arctan(x)) = \frac{1}{1+x^2}$$

and thus that

$$g'(x) = \frac{2}{1+x^2}.$$

As well: by using the above triangles, notice that

$$\begin{aligned}
\sin(g(x)) &= \sin(2\arctan(x)) \\
&= 2\sin(\arctan(x))\cos(\arctan(x)) \\
&= 2 \cdot \frac{1}{\sqrt{1+x^2}} \cdot \frac{x}{\sqrt{1+x^2}} \\
&= \frac{2x}{1+x^2},
\end{aligned}$$

and

$$\begin{aligned}
\cos(g(x)) &= \cos(2\arctan(x)) \\
&= 2\cos^2(\arctan(x)) - 1 \\
&= \frac{2}{1+x^2} - 1 \\
&= \frac{1-x^2}{1+x^2}.
\end{aligned}$$

Finally, note that trivially we have that

$$g^{-1}(x) = \tan(x/2),$$

by definition.

What does this all mean? Well: suppose we have some function $f(x)$ where all of its terms are trig functions – i.e. $f(x) = \frac{1}{1+\sin(x)}$, or $f(x) = \frac{1}{\cos(x)}$ – and we make the substiution

$$\int_a^b f(x) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x))g'(x), \quad \text{for } g(x) = 2\arctan(x).$$

What do we know about the integral on the right? Well: as we've just shown above, the substitution of $g(x)$ turns all of the $\sin(x)$'s into $\sin(g(x))$'s, which are just reciprocals of polynomials; similarly, we've turned all of the $\cos(x)$'s into $\cos(g(x))$'s, which are also made of polynomials. In other words, this substitution turns a function that's made entirely out of trig functions into one that's made **only out of polynomials!** – i.e. it turns trig functions into quadratic polynomials! This is excellent for us, because (as you may have noticed) it's often *far* easier to integrate polynomials than trig functions.

This substitution is probably one of those things that's perhaps clearer in its use than its explanation. We provide an example here:

5

Example 2.2. Find the integral

$$\int_0^{\pi/2} \frac{1}{1 + \sin(\theta)} d\theta.$$

Proof. So: without thinking, let's just try our substitution $\theta = g(x)$, where $g(x) = 2\arctan(x)$:

$$\begin{aligned}
\int_0^{\pi/2} \frac{1}{1 + \sin(\theta)} d\theta &= \int_{g^{-1}(0)}^{g^{-1}(\pi/2)} f(g(x))g'(x)dx \\
&= \int_{\tan(0)}^{\tan(\pi/4)} \frac{1}{1 + \frac{2x}{1+x^2}} \cdot \frac{2}{1+x^2} dx \\
&= \int_0^1 \frac{2}{1 + x^2 + 2x} dx \\
&= \int_0^1 \frac{2}{(1+x)^2} dx \\
&= \int_1^2 \frac{2}{x^2} dx \\
&= \left. -\frac{2}{x} \right|_1^2 \\
&= 1/2.
\end{aligned}$$

... so it works! Without any effort, we were able to just mechanically calculate an integral that otherwise looked quite impossible. Neat! $\qquad\square$

2.3 The Gaussian Integral

Finally, we close with one last particularly esoteric integral, which shows up on your HW this week:

$$\int e^{-x^2} dx.$$

This is the **Gaussian integral**: it shows up in pretty much every branch of the sciences, largely because the function e^{-x^2} corresponds to the normal distribution. Surprisingly/frustratingly, for an integral that shows up in so many places in the sciences, it can be proven[1] that there is **no way to express this integral in terms of the elementary functions** $\sin, \cos, \tan, e^x, \log$, polynomials, and their various combinations.

This, understandably, makes calculating this integral rather difficult. Typically, when we want to integrate something, like (say) $\cos(x)$, we find an antiderivative for our function first: i.e.

$$\int \cos(x)dx = \sin(x) + C,$$

---

[1]Via the Risch algorithm.

and use this antiderivative to calculate definite integrals: i.e.

$$\int_0^\pi \cos(x)dx = \sin(\pi) + C - \sin(0) - C = 0 + C - 0 - C = 0.$$

Frustratingly, e^{-x^2}, as we said above, has no such "simple" antiderivative. However, despite this, we can still integrate it over certain regions! – in particular, we can integrate this function over all of $\mathbb{R}$, where it turns out that

$$\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}.$$

The techniques to do this are rather tricky: we outline them here.

First, we need a tool that usually comes up in multivariable calculus, but which we can actually prove using entirely single-variable techniques:

Theorem 2.3. (Fubini) Let $f(u,v)$ be a function $\mathbb{R}^2 \to \mathbb{R}$: i.e. a function that takes in two real numbers and returns a real number. Suppppose that for any fixed value x of u, the function $f(x,v)$, thought of a single-variable function in y, is continuous; similarly, for any fixed value y of v, suppose that the function $f(u,y)$ is also continuous. Furthermore, suppose that the function $f(u,v)$ is bounded.

For a fixed value of x, because $f(x,v)$ is a bounded and continuous function of one variable v, we can calculate the integral

$$\int_0^y f(x,v)dv;$$

furthermore, by the fundamental theorem of calculus, we know that the function

$$F(y) = \int_0^y f(x,v)dv$$

is a continuous and bounded function of y on any finite interval.

Similarly, for any fixed value of y, we can calculate the function

$$G(y) = \int_0^x f(u,y)du,$$

which is also continuous and bounded on any finite interval.

Justified by these comments, we know that the two integrals

$$\int_0^x \left(\int_0^y f(u,v)dv \right) du, \int_0^y \left(\int_0^x f(u,v)du \right) dv$$

exist.

Fubini's theorem is the claim that these two integrals are the **same**. In other words, that if we integrate a function in two variables once in each variable, it doesn't matter in what order we do those integrations: integrating with respect to x and then y is the same as integrating with respect to y and then x.

We omit the proof here; it's something that you can do using only single-variable calculus tricks (specifically, the fundamental theorem of calculus, plus the definitions of the derivative and the integral,) but it's ugly and not really what we want to focus on with this lecture.

The applications of this theorem, however, *are* what we want to focus on. Specifically: we were studying the integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx.$$

First, notice that the integral of this function exists. Why? Well: notice that the function e^{-x^2} is bounded above by the function $e^{-|x|}$ for all values of $x \geq 1$. The area under the curve of $e^{-|x|}$ from 1 to infinity is finite, as it's just the integral

$$\int_{1}^{\infty} e^{-x} = -e^{-x} \Big|_{1}^{\infty} = e^{-1}.$$

Therefore, the area under the curve e^{-x^2} is finite: this is because it's bounded above by 1 on the interval $[-1, 1]$ and by $e^{-|x|}$ everywhere else, and the total area underneath these two curves is just $2 + \frac{2}{e}$. Therefore, we can measure the area under this curve! Call this quantity I: then, we have that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = I.$$

Because our function e^{-x^2} is symmetric across the y-axis, we can simplify this a bit to the expression

$$2 \int_{0}^{\infty} e^{-x^2} dx = I.$$

So: how can we integrate this? Intuitively, what we'd *like* to do is use integration by substitution, with $u = -x^2$. However, there is no $-2xdx$ in the rest of the integral, which means that we can't just immediately apply this substitution: we need to find a way to somehow multiply our function by x.

To do this, we do something really clever and surprising: square both sides! This gives us

$$I^2 = 4 \left(\int_{0}^{\infty} e^{-x^2} dx \right)^2$$

8

Why is this useful? Well, notice that we can write

$$I^2 = 4 \left(\int_0^\infty e^{-x^2} dx \right)^2$$

$$= 4 \left(\int_0^\infty e^{-x^2} dx \right) \cdot \left(\int_0^\infty e^{-x^2} dx \right)$$

$$= 4 \left(\int_0^\infty e^{-x^2} dx \right) \cdot \left(\int_0^\infty e^{-y^2} dy \right).$$

Notice that in the third line we replaced the variable x with a y: this is OK, because integrals don't care what letter you're using for your variable (i.e. $\int f(x)dx = \int f(y)dy$: the x and y are just letters, nothing more.)

So: with respect to our first integral, the second integral is just a constant. Therefore, we can pull this constant $\left(\int_{-\infty}^\infty e^{-y^2} dy \right)$ into our first integral: i.e. we have

$$I^2 = 4 \int_0^\infty e^{-x^2} \cdot \left(\int_0^\infty e^{-y^2} dy \right) dx.$$

Similarly, with respect to the dy-integral, the expression e^{-x^2} is just a constant: the inner integral only cares about y's. Therefore, we can pull the e^{-x^2} term inside of the second integral, which gives us

$$I^2 = 4 \int_0^\infty \left(\int_0^\infty e^{-x^2} \cdot e^{-y^2} dy \right) dx.$$

Combining the two e-terms into $e^{-(x^2+y^2)}$ gives us

$$I^2 = 4 \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)} dy \right) dx.$$

Now that we have written our integral in this form, we can finally see a trick we could use to get a x-term into our function: perform the substitution $y = xv$. Within the integral with respect to y, x is just a constant: so we have that $dy = xdv$, and therefore that our integral is

$$4 \int_0^\infty \left(\int_0^\infty e^{-(x^2+x^2v^2)} xdv \right) dx = 4 \int_0^\infty \left(\int_0^\infty xe^{-x^2(1+v^2)} dv \right) dx$$

We now have a x-term multiplied by our function! However, we're not integrating with respect to x at the moment: instead, we're integrating with respect to v.

To fix this, use Fubini's theorem to switch the order of integration! This gives us the integral

$$4 \int_0^\infty \left(\int_0^\infty x e^{-x^2(1+v^2)} dx \right) dv,$$

which we *can* now evaluate! Specifically, let $u = x^2(1 + v^2)$: then $du = 2x(1 + v^2)dx$, because v is a constant with respect to x, and $x dx = \frac{1}{2} \frac{du}{(1+v^2)}$.

Plugging this in gives us

$$4 \int_0^\infty \left(\int_0^\infty \frac{1}{2} \frac{1}{(1 + v^2)} e^{-u} du \right) dv,$$

which (by pulling the term $\frac{1}{2} \frac{1}{(2(1+v^2)}$ out of the du-integral) is just

$$2 \int_0^\infty \frac{1}{1 + v^2} \left(\int_0^\infty e^{-u} du \right) dv = 2 \left(\int_0^\infty \frac{1}{1 + v^2} \right) \cdot \left(\int_0^\infty e^{-u} du \right)$$

We can integrate both of these functions! Doing so (left to you) will yield the value π, which tells us that $I^2 = \pi$: i.e. that $I = \sqrt{\pi}$. (We know I isn't $-\sqrt{\pi}$, the other possible square root of π, because e^{-x^2} is a positive function.)

3 Polar and Parametric Integrals

3.1 Polar Coördinates and Integration: Theory

Typically, when we discuss points in the plane, we use **Cartesian coördinates**, where we refer to a point in the plane as a pair (x, y), where x denotes its horizontal location along the x-axis and y denotes its vertical location along the y-axis:
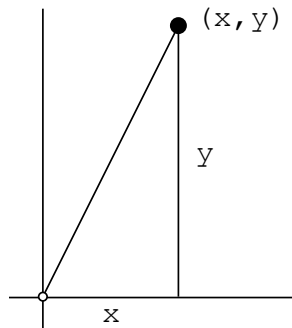
This, however, is not the only way to label the points in the plane! Specifically, another choice of coördinates you could make is that of the **polar coördinate system**, where we label every point in the plane as a pair (r, θ), where r denotes the distance of this point from the origin and θ denotes the angle made by the positive x-axis and the vector represented by our point, measured counterclockwise:
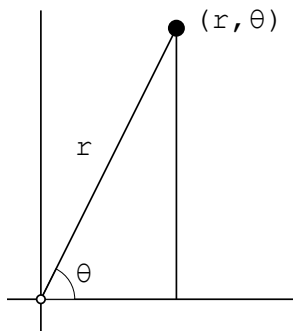
In this fashion, we can imagine the plane as the collection of all pairs of points $[0, 2\pi) \times \mathbb{R}^+$, and can study the graphs of functions $f(\theta)$ that map various angles to radial values. Can we integrate these functions?
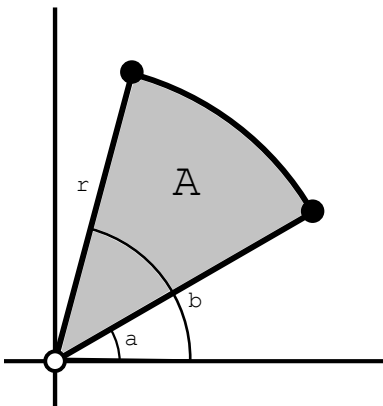
Well: for cartesian-coördinate-functions, we started by integrating step functions and used this to develop a theory of integration for the other functions. Can we do the same here?

First, let's say what a polar step function is:

Definition 3.1. A **polar step function** is a map $f : [a, b] \to \mathbb{R}$ with a partition $a = t_0 < t_1 < \ldots t_n = b$ of $[a, b]$, such that f is constant on each of the open intervals (t_{i-1}, t_i).
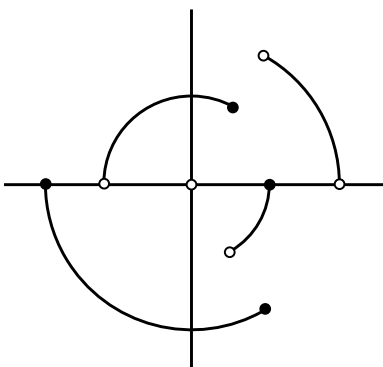
What is the area bounded by such a function? Well: first, recall that the area of a $(b-a)$-radian "wedge" of a circle with radius r is just $r^2 \cdot \frac{(b-a)}{2}$, as it's just $\frac{b-a}{2\pi}$-th of a circle with area πr^2.

11

Consequently, for a polar step function φ, we can say that the area radially bounded above by φ over $[a, b]$ is

$$\text{area}(\varphi, [a, b]) = \sum_{i=1}^{n} \left(\text{value}_{(t_{i-1}, t_1)}(\varphi(\theta))\right)^2 \cdot \frac{(t_{i-1}, t_i)}{2}$$

But this expression is just the "normal" integral – i.e. integral of φ as treated as a cartesian-coordinate-function– of $(\varphi(\theta))^2/2$: i.e.

$$\text{area}(\varphi, [a, b]) = \int_a^b \frac{\varphi(\theta))^2}{2} d\theta.$$

Does this formula hold in general? As it turns out, yes! Specifically, we have the following theorem:

Theorem 3.2. If f is a polar function such that $\text{area}(f, [a, b])$ is well-defined, then

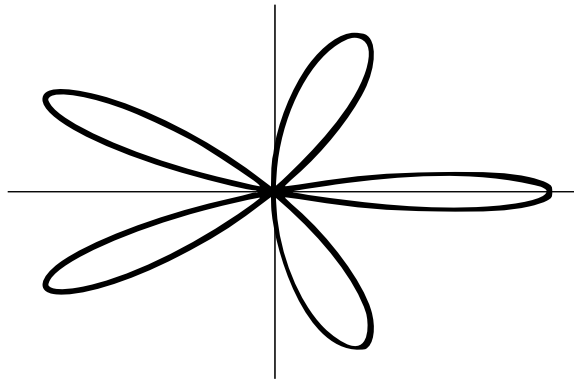$$\text{area}(f, [a, b]) = \int_a^b \frac{(f(\theta))^2}{2} d\theta.$$

3.2   Polar Coördinates and Integration: Examples

To illustrate how these formulas are used, we work a few examples here.

Example 3.3. Find the area contained by the polar rose

$$f(\theta) = \cos(5\theta)$$

over the domain $[0, \pi]$.
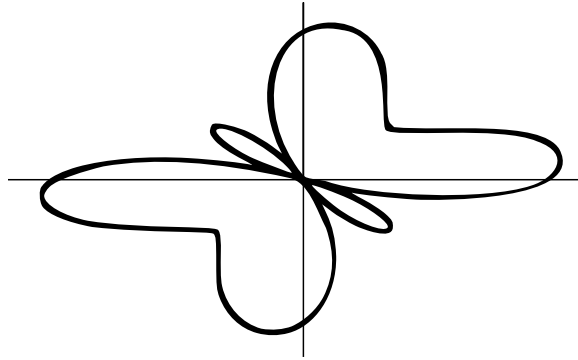
Proof. By our above formula, we know that

$$\text{area(rose)} = \int_0^\pi \frac{\cos^2(5\theta)}{2} d\theta$$

$$= \int_0^\pi \frac{1 + \cos(10\theta)}{4} d\theta \qquad \text{(double-angle formula)}$$

$$= \frac{10\theta + \sin(10\theta)}{40} \Bigg|_0^\pi$$

$$= \frac{10\pi + 0}{40} - \frac{10 \cdot 0 + 0}{40}$$

$$= \pi/4.$$

$\square$

Example 3.4. Find the area contained within the "butterfly curve"

$$f(\theta) = 1 + \cos(4\theta) + \sin(2\theta)$$

over the domain $[0, 2\pi)$.

Proof. Again, we simply use our earlier formula and calculate, liberally using the double-angle formulas:

$$\text{area(butterfly)} = \int_0^{2\pi} \frac{(1 + \cos(4\theta + \sin(2\theta))^2}{2} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1 + \cos^2(4\theta) + \sin^2(2\theta) + 2\cos(4\theta) + 2\sin(2\theta) + 2\cos(4\theta)\sin(2\theta)\right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1 + \frac{1 + \cos(8\theta)}{2} + \frac{1 - \cos(4\theta)}{2} + 2\cos(4\theta) + 2\sin(2\theta) + 2\left(2\cos^2(2\theta) - 1\right)\sin(2\theta)\right) d\theta$$

$$= \left(\frac{\theta}{2} + \frac{8\theta + \sin(8\theta)}{32} + \frac{4 - \sin(4\theta)}{16} + \frac{\sin(4\theta)}{4} - \frac{\cos(2\theta)}{2}\right)\Bigg|_0^{2\pi} + \int_0^{2\pi} \left(\left(2\cos^2(2\theta) - 1\right)\sin(2\theta)\right) d\theta$$

$$= 2\pi + \int_0^{2\pi} \left(\left(2\cos^2(2\theta) - 1\right)\sin(2\theta)\right) d\theta.$$

13

Finally, to evaluate the last integral, perform a u-substitution with $u = \cos(2\theta)$, $u' = 2\sin(2\theta)$:

$$
\begin{aligned}
\text{area(butterfly)} &= 2\pi + \int_0^{2\pi} \left(\left(2\cos^2(\theta) - 1 \right) \sin(2\theta) \right) d\theta \\
&= 2\pi + \int_{\cos(0)}^{\cos(4\pi)} \frac{2u^2 - 1}{2} du \\
&= 2\pi + \int_1^1 \frac{2u^2 - 1}{2} du \\
&= 2\pi.
\end{aligned}
$$

□

3.3   Parametrization and Integration: Theory

As we've discussed above, there are ways to draw curves in $\mathbb{R}^2$ other than using Cartesian coördinates: polar coördinates, as we've just discussed, provide a remarkably useful way to create simple curves. Are there other methods for drawing curves, that might perhaps also come in handy?

As it turns out, yes! Specifically, to draw a curve c in the plane, it suffices to offer a pair of functions $x(t)$ and $y(t) : \mathbb{R} \to \mathbb{R}$, and look at the graph of $c(t) = (x(t), y(t))$ on some interval $[a, b]$. For example, if we let

$$
\begin{aligned}
x(t) &= t, \\
y(t) &= t^2,
\end{aligned}
$$

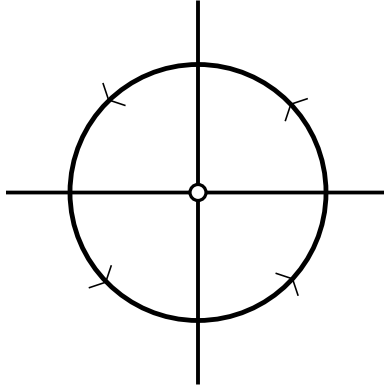we get the graph of the parabola sketched below:
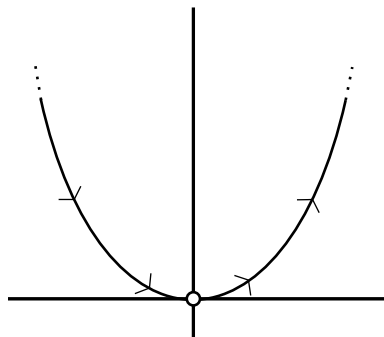
The arrows above indicate the path of the curve as t increases: i.e. small increases in the value of t follow the arrows drawn on the graph above.

Another example is the unit circle, given by the curve

$$
\begin{aligned}
x(t) &= \cos(t), \\
y(t) &= \sin(t),
\end{aligned}
$$

on the interval $[0, 2\pi)$:

14

here, the curve starts at $(1, 0)$ and spirals counterclockwise, as drawn.

This process of describing a curve by a pair of equations is known as **parametrization**. A natural question to ask, then, is the following: can we come up with an expression for the "area" beneath a parametrized curve?

So: one idea we could use is the following:

Definition 3.5. For a parametric curve $c(t) = (x(t), y(t))$, we say that the area beneath $c(t)$ on some interval $[a, b]$ is well-defined if and only if for every increasingly fine[2] set of partitions P_n of $[a, b]$, we have that

$$\lim_{n \to \infty} \left(\sum_{P_n} \sup_{(t_{i-1}, t_i)} (y(t)) \cdot (x(t_i) - x(t_{i-1})) \right) = \lim_{n \to \infty} \left(\sum_{P_n} \inf_{(t_{i-1}, t_i)} (y(t)) \cdot (x(t_i) - x(t_{i-1})) \right).$$

Pictorially, this just means that for any collection of incrasingly fine partitions P_n, the area of the red rectangles below (upper sums) converges to the area of the blue rectangles below (lower sums):

---

[2]A set of partitions is called **increasingly fine** if for every $\epsilon > 0$, there's a N so that for any $n > N$, we have that all of the distances $t_i - t_{i-1}$ are $< \epsilon$ in the partition $P_n = t_0 < \ldots < t_n$. In other words, increasingly fine partitions have arbitrarily tiny step sizes.
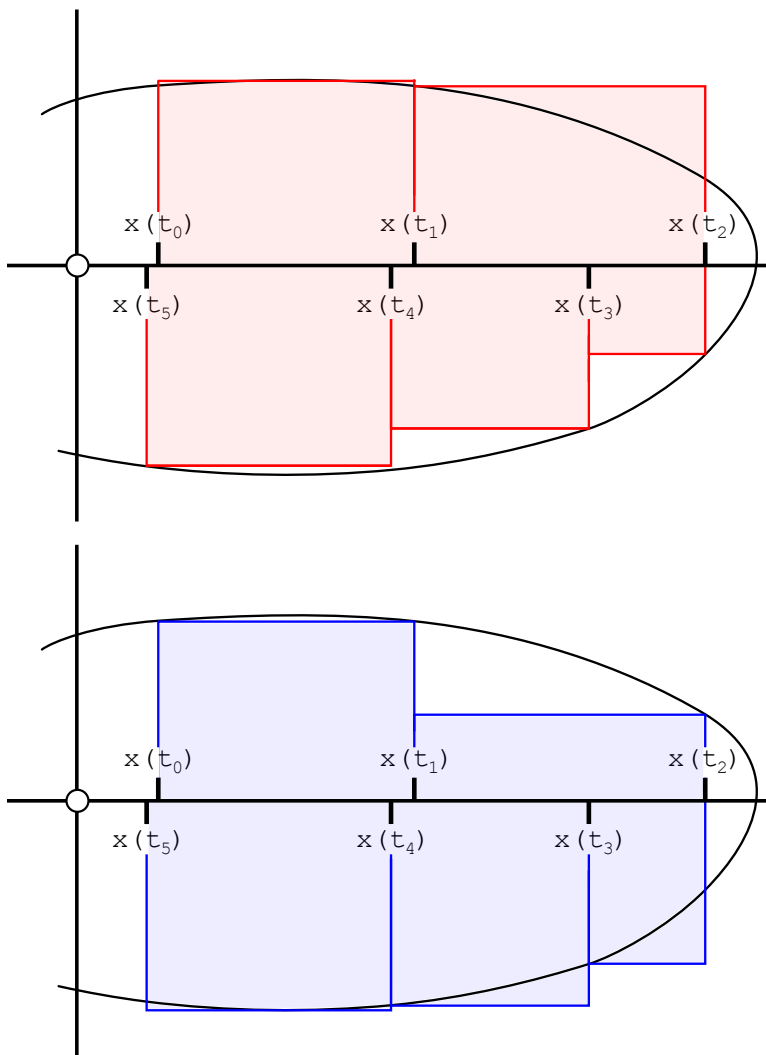
Given this definition, we can make the following observation:

Theorem 3.6. If $c(t) = (x(t), y(t))$ is a parametrized curve such that the area beneath $c(t)$ on some interval $[a, b]$ is well-defined, $y(t)$ is continuous and bounded on $[a, b]$, and $x(t)$ is C^1 and bounded with bounded derivative on $[a, b]$, then
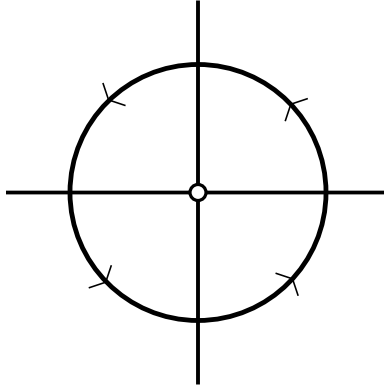
$$\text{area}(c(t), [a, b]) = \int_a^b y(t) x'(t) dt.$$

The next section features several examples of how this formula works, and several caveats about its use:

3.4 Parametrization and Integration: Examples

Example 3.7. Find the unsigned area enclosed by the unit circle

$$c(t) = (\cos(t), \sin(t)).$$

16

Proof. If we apply this formula in a straightforward manner, we get

$$\text{area}(c(t), [0, 2\pi]) = \int_0^{2\pi} \sin(t) \cdot (-\sin(t))dt$$

$$= -\int_0^{2\pi} \sin^2(t)dt$$

$$= -\int_0^{2\pi} \frac{\cos(2t) - 1}{2}dt$$

$$= \frac{\sin(2t) - 2t}{4}\bigg|_0^{2\pi}$$

$$= -\pi.$$

... wait. Where did that minus sign come from?

First: remember that our integrals are computing, typically, an idea of "signed area:" i.e. area beneath the x-axis is typically counted negatively, whereas area above the x-axis is counted positively. Well, as it turns out, the **orientation** of our parametrized curve *also* affects the sign of the area! Specifically, parametrized area is counted negatively when the curve's x-coördinates are decreasing, and positively when the curve's x-coördinates are increasing: you can see this in the derivation of our formula for the area of a parametric curve, as the sign of the sum

$$\lim_{n \to \infty} \left(\sum_{P_n} \sup_{(t_{i-1}, t_i)} (y(t)) \cdot (x(t_i) - x(t_{i-1})) \right)$$

depends both on the sign of $y(t)$ **and** of $(x(t_i) - x(t_{i-1}))$. So the curve above has *signed* area $-\pi$, but *unsigned* area π. □