Math 8	Instructor: Padraic Bartlett

Practice Final: Solutions

Week 10	*Caltech 2012*

1. Determine whether the following series converge:

(a)

$$\sum_{n=1}^{\infty} \frac{1}{(\ln(n))^k)}.$$

Solution. First, notice that because $\ln(n)$ is a positive and monotonically increasing function on $[1, \infty)$, the function $\frac{1}{(\ln(n))^k}$ is a positive and monotonically decreasing function on $[1, \infty)$. Therefore, by the integral test, we know that our series converges if and only if the improper integral

$$\int_1^{\infty} \frac{1}{(\ln(x))^k} dx$$

exists and is finite. We evaluate this integral via the u-substitution $u = \ln(x) \Rightarrow x = e^u$, $du = \frac{1}{x} dx \Rightarrow dx = e^u du$:

$$\begin{aligned} \int_1^{\infty} \frac{1}{(\ln(x))^k} dx &= \int_{\ln(1)}^{\lim_{a\to\infty} \ln(a)} \frac{1}{u^k} \cdot e^u du \\ &= \int_0^{\infty} \frac{e^u}{u^k} du. \end{aligned}$$

Notice that $\lim_{u\to\infty} \frac{e^u}{u^k} = \infty$; you can see this by applying L'Hôpital's theorem k times, at which point the limit will become $\lim_{u\to\infty} \frac{e^u}{k!}$. (This is justified because at every step up to k, the top was e^u and the bottom was a nonconstant polynomial.) Therefore, because the function we're integrating goes off to infinity as $u \to \infty$, its integral on $[0, \infty]$ must diverge to infinity as well.

(b)

$$\sum_{n=1}^{\infty} \frac{1}{(\ln(n))^n}.$$

Solution. We use a similar argument to (a.) Again, because $\ln(n)$ is a positive and monotonically increasing function on $[1, \infty)$, the function $\frac{1}{(\ln(n))^n}$ is a positive and monotonically decreasing function on $[1, \infty)$. Therefore, by the integral test, we know that our series converges if and only if the improper integral

$$\int_1^{\infty} \frac{1}{(\ln(x))^x} dx$$

exists and is finite. We evaluate this integral via the same u-substitution $u = \ln(x) \Rightarrow x = e^u$, $du = \frac{1}{x}dx \Rightarrow dx = e^u du$ as before:

$$\begin{aligned}\int_1^\infty \frac{1}{(\ln(x))^x}dx &= \int_{\ln(1)}^{\lim_{a\to\infty}\ln(a)} \frac{1}{u^{e^u}} \cdot e^u du \\ &= \int_0^\infty \frac{e^u}{u^{e^u}} du \\ &= \int_0^\infty \frac{e^u}{e^{\ln(u)\cdot e^u}} du \\ &= \int_0^\infty e^{u-e^u \ln(u)} du\end{aligned}$$

We want to know whether this integral is finite or not. By itself, we don't know: it looks hard to directly integrate! However, there are simpler upper bounds for the function we're integrating that are easier to deal with: so let's do that instead.

Notice that for $u > 3$, we have that $u - e^u \ln(u) < -u$. This is because it's true at $u = 3$ (because $3 - e^3 \ln(3) \sim -18.6 < -3$) and the derivative of $u - e^u \ln(u)$, $1 - \frac{e^u}{u} - e^u \ln(u)$, is less than $1 - e^u \ln(u)$, which is less than $1 - e^u$ for $u > 3$, which is less than -1, the derivative of $-u$, again for $u > 3$. Therefore, for $u > 3$, we have $u - e^u \ln(u) < u - e^u < -u$; so we've shown that

$$e^{u-e^u \ln(u)} < e^{-u}.$$

So: how can we relate the integral of e^{-u} to the integral of $e^{u-e^u \ln(u)}$? Easy: use the integral test again!

In particular, notice that the series $\sum_{n=1}^\infty e^{-n}$ converges by the ratio test ($\lim_{n\to\infty} \frac{e^{-(n+1)}}{e^{-n}} = \frac{1}{e} < 1$). Therefore, by the comparison test, the series

$$\sum_{n=1}^\infty e^{n-e^n \ln(n)}$$

must also converge.

But as we showed above, the derivative of $u - e^u \ln(u)$ is < -1 for $u > 3$, so $e^{u-e^u \ln(u)}$ is a monotonically decreasing function for $u > 3$, and is therefore an eventually monotonically decreasing function. Therefore, because the series $\sum_{n=1}^\infty e^{n-e^n \ln(n)}$ converges, the corresponding integral

$$\int_0^\infty e^{u-e^u \ln(u)} du$$

converges, by our second application of the integral test. But this means that our original series

$$\sum_{n=1}^\infty \frac{1}{(\ln(n))^n}$$

also converges, by our **first** application of the integral test!

(c)

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n}.$$

Solution. We proceed by the comparison test. The idea is the following: we know that $\sin(x) \leq x$, for any positive value of x. (To see why: it's true at $x = 0$. As well, the derivative of x is 1, which is always greater than the derivative of $\sin(x)$, which is $\cos(x)$. Therefore, going forward, we have $\sin(x) < x$, for all positive x.) We seek to apply the comparison test here. We can do so because $\sin(1/n)$ is always positive (because $\sin(x)$ is positive on $[0, \pi]$). If we compare $\sin(1/n)$ to $1/n$, we have that the series in (c) converges if the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. We've proven in class that this series converges; therefore, our original series also converges.

2. Evaluate the improper integral

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx.$$

Solution. Try the u-substitution $u = \sqrt{x^2-1} \Rightarrow x = \sqrt{u^2+1}$. If you do this, you get that $du = \frac{x}{\sqrt{x^2-1}} dx \Rightarrow \frac{u}{\sqrt{u^2+1}} du = dx$, and therefore that our original integral is

$$\int_{\sqrt{2^2-1}}^{\lim_{a\to\infty} \sqrt{a^2-1}} \frac{1}{u\sqrt{u^2+1}} \cdot \frac{u}{\sqrt{u^2+1}} du = \int_{\sqrt{3}}^{\infty} \frac{1}{u^2+1} du.$$

Now, you should try a trig substitution! In particular, try $u = \tan(t), t = \arctan(u), du = \frac{1}{\cos^2(t)} dt$:

$$\begin{aligned}
\int_{\sqrt{3}}^{\infty} \frac{1}{u^2+1} du &= \int_{\arctan(\sqrt{3})}^{\lim_{a\to\infty} \arctan(a)} \frac{1}{1+\tan^2(u)} \cdot \frac{1}{\cos^2(u)} du \\
&= \int_{\arctan(\sqrt{3})}^{\lim_{a\to\infty} \arctan(a)} 1 du \\
&= \left(\lim_{a\to\infty} \arctan(a)\right) - \arctan(\sqrt{3}).
\end{aligned}$$

We know that tangent approaches positive-infinity on $(-\pi/2, \pi/2)$ as its argument approaches $\pi/2$: therefore, the limit as arctangent approaches $+\infty$ is just $\pi/2$. Similarly, we know that tangent is equal to $\sqrt{3}$ when its argument is equal to $\pi/3$; therefore, $\arctan(\sqrt{3})$ is $\pi/3$. Therefore, our integral is just is $\pi/6$.

3. Use Taylor polynomials to approximate $\sin(.8)$ to within $\pm 10^{-4}$.
Solution. Recall that the $2n+1$-degree Taylor polynomial for $\sin(x)$ around 0 is just

$$T_{2n+1}(\sin(x);0) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Write $\sin(x)$ as the sum of its $2n+1$-degree Taylor polynomial and its $2n+1$-degree remainder function:

$$\sin(x) = T_{2n+1}(\sin(x);0) + R_{2n+1}(\sin(x);0).$$

If we can make $|R_{2n+1}(\sin(x);0)| < 10^{-5}$, for some value of n, we can then approximate $\sin(x)$ by using its corresponding Taylor polynomial.

So: Taylor's theorem says that for any $x > 0$, we have

$$|R_{2n+1}(\sin(x);0)| = \left| \int_0^x \frac{\left(\frac{d^{(2n+1)}}{dy^{(2n+1)}}(\sin(y))\right)\Big|_{y=t}}{(2n+1)!}(x-t)^{(2n+1)}dt \right|.$$

We only want to show that this is small; so we can bound various things in this integral above by other values. In particular, any derivative of sin will be ≤ 1 in terms of magnitude, so we can replace the derivatives with 1; as well, we can replace the quantity $(x-t)^{2n+1}$ with x^{2n+1}. This gives us

$$|R_{2n+1}(\sin(x);0)| \leq \left| \int_0^x \frac{x^{(2n+1)}}{(2n+1)!}dt \right| = \frac{x^{2n+2}}{(2n+1)!}.$$

For $x = .8$, this is $\leq 10^{-4}$ for the first time at $n = 3$.

Therefore, $\sin(.8)$ is equal to $T_{2\cdot 3+1}(\sin(x);0)$ at .8, to within $\pm 10^{-4}$, and therefore is roughly

$$T_7(\sin(x);0)\Big|_{x=.8} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)\Big|_{x=.8} \sim .71736$$

to within $\pm 10^{-4}$.

4. (a) Find the Taylor series for $\ln(1+x^6)$ centered around 0.
Solution. First, recall that the Taylor series for $\ln(1-x)$ was

$$\sum_{n=1}^{\infty} -\frac{x^n}{n},$$

which was valid for $x \in (-1,1)$: in other words, for any $x \in (-1,1)$, we had

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}.$$

If we plug in $-x^6$ for x in the above expression, we get

$$\ln(1+x^6) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{6n}}{n},$$

which is again true for all $x \in (-1,1)$. Because Taylor series are the unique power series representation of a function where they exist, and $\ln(1+x^6)$ is an infinitely differentiable function on $(-1,1)$, its Taylor series must be

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{6n}}{n}.$$

(b) Using the power series above, what complex power series would you use to define $f(z) = \ln(1+x^6)$ in the complex plane?
Solution. Just like we did in class when we defined e^z, we might try

$$\ln(1+z^6) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{6n}}{n}.$$

(c) What is the radius of convergence R of this power series?
Solution. Take any real value of $x > 0$. Then, because x is real and positive, we can use the ratio test to see that the series

$$\sum_{n=1}^{\infty} \frac{x^{6n}}{n}$$

converges when

$$\lim_{n\to\infty} \frac{x^{6(n+1)}/(n+1)}{x^{6n}/n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot x^6 = x^6$$

is less than 1. In other words, this series converges for positive real values of $x < 1$.
Because absolute convergence implies convergence, this means that the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{6n}}{n}.$$

also converges when z is real and in $[0,1)$. Therefore, by our theorem on radii of convergence, our series must converge for **any** $z \in \mathbb{C}$ with $||z|| < 1$.
However, we can also see that this series diverges when $z = i$, because

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{((i)^6)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, our series diverges for at least one value of z with magnitude 1. Consequently, because our series converges for any z with $||z|| < 1$, and diverges for a value of z with $||z|| = 1$, the radius of convergence of our series must be **exactly** 1.

(d) Find two values of $z \in \mathbb{C}$ with $||z|| = R$ such that $f(z)$ converges, and two more values of $z \in \mathbb{C}, ||z|| = R$ such that $f(z)$ diverges.

Solution. If $z = \pm 1$, then our series is just

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1^{6n}}{n},$$

which is the alternating harmonic series (which converges.)

However, if $z = \pm e^{i\pi/6}$, then $z^6 = e^{i\pi} = -1$, and therefore our series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges (because it's -1 times the harmonic series.)