Math 8	Instructor: Padraic Bartlett
Lecture 10: Complex Numbers and Series	
Week 10	*Caltech 2012*

1 The Structure of the Complex Numbers

Earlier in Ma1a, we often ran into the following question: "Given some polynomial $P(x)$, what are its roots?" Depending on the polynomial, we had several techniques for finding these roots (Rolle's theorem, quadratic/cubic formulas, factorization;) however, we at times would encounter polynomials that have no roots at all, like

$$x^2 + 1.$$

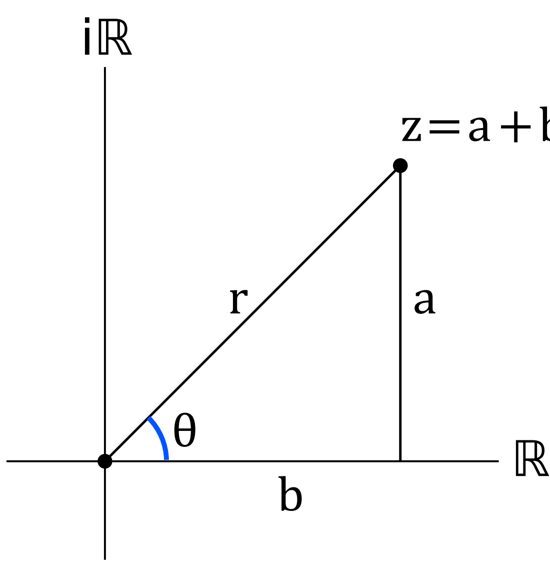
Yet, despite the observation that this polynomial's graph never crossed the x-axis, we *could* use the quadratic formula to find that this polynomial had the "formal" roots

$$\frac{-0 \pm \sqrt{-4}}{2} = \pm\sqrt{-1}.$$

The number $\sqrt{-1}$, unfortunately, isn't a real number (because $x^2 \geq 0$ for any real x, as we proved last quarter) – so we had that this polynomial has no roots over $\mathbb{R}$. This was a rather frustrating block to run into; often, we like to factor polynomials entirely into their roots, and it would be quite nice if we could always do so, as opposed to having to worry about irreducible functions like $x^2 + 1$.

Motivated by this, we can create the **complex numbers** by just throwing $\sqrt{-1}$ into the real numbers. Formally, we define the set of complex numbers, $\mathbb{C}$, as the set of all numbers $\{a + bi : a, b \in \mathbb{R}\}$, where $i = \sqrt{-1}$.

Graphically, we can visualize the complex numbers as a plane, where we identify one axis with the real line $\mathbb{R}$, the other axis with the imaginary-real line $i\mathbb{R}$, and map the point $a + bi$ to (a, b):



Two useful concepts when working in the complex plane are the ideas of **norm** and **conjugate**:

Definition. If $z = x + iy$ is a complex number, then we define $|z|$, the **norm** of z, to be the distance from z to the origin in our graphical representation; i.e. $|z| = \sqrt{x^2 + y^2}$.

As well, we define the **conjugate** of $z = x + iy$ to be the complex number $\overline{z} = x - iy$. Notice that $|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}$.

In the real line, recall that we had $|x \cdot y| = |x| \cdot |y|$; this still holds in the complex plane! In particular, we have $|w \cdot z| = |w| \cdot |z|$, for any pair of complex numbers w, z. (If you don't believe this, prove it! – it's not a difficult exercise to check.)

So: we have this set, $\mathbb{C}$, that looks like the real numbers with i thrown in. Do we have any way of extending any of the functions we know and like on $\mathbb{R}$, like $\sin(x), \cos(x), e^x$ to the complex plane?

At first glance, it doesn't seem likely: i.e. what should we say $\sin(i)$ is? Is cos a periodic function when we add multiples of $2\pi i$ to its input? Initially, these questions seem unanswerable; so (as mathematicians often do when faced with difficult questions) let's try something easier instead!

In other words, let's look at **polynomials**. These functions are much easier to extend to $\mathbb{C}$: i.e. if we have a polynomial on the real line

$$f(x) = 2x^3 - 3x + 1,$$

the natural way to extend this to the complex line is just to replace the x's with z's: i.e.

$$f(z) = 2z^3 - 3z + 1.$$

This gives you a well-defined function on the complex numbers (i.e. you put a complex number in and you get a complex number out,) such that if you restrict your inputs to the real line $x + i \cdot 0$ in the complex numbers, you get the same outputs as the real-valued polynomial.

In other words, we know how to work with polynomials. Does this help us work with more general functions? As we've seen over the last two weeks, the answer here is yes! More specifically, the answer here is to use **power series**. Specifically, over the last week, we showed that

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots, \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots, \text{and} \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots
\end{aligned}$$

for all real values of x. Therefore, we can choose to **define**

$$\begin{aligned}
\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots, \\
\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots, \text{and} \\
e^z &= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots,
\end{aligned}$$

for all $z \in \mathbb{C}$. This extension has the same properties as the one we chose for polynomials: it gives a nice, consistent definition of each of these functions over all of $\mathbb{C}$, that agrees with the definitions they already had on the real line $\mathbb{R}$.

The only issue with these extensions is that we're still not entirely quite sure what they mean. I.e.: what **is** $\sin(i)$, apart from some strange infinite power series? Where does the point e^z lie on the complex plane?

To answer these questions, let's look at e^z first, as it's arguably the easiest of the three (its terms don't do the strange alternating-thing, and behave well under most algebraic manipulations.) In particular, write $z = x + iy$: then we have

$$e^z = e^{x+iy} = e^x \cdot e^{iy},$$

where e^x is just the real-valued function we already understand. So, it suffices to understand e^{iy}, which we study here:

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \frac{(iy)^7}{7!} + \frac{(iy)^8}{8!} + \dots$$

If we use the fact that $i^2 = -1$, we can see that powers of i follow the form

$$i, -1, -i, 1, i, -1, -i, 1, \dots$$

and therefore that

$$e^{iy} = 1 + iy - \frac{y^2}{2} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \frac{y^6}{6!} - i\frac{y^7}{7!} + \frac{y^8}{8!} + \dots$$

If we split this into its real and imaginary parts, we can see that

$$e^{iy} = \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} \dots\right).$$

But wait! We've seen those two series before: they're just the series for $\sin(y)$ and $\cos(y)$! In other words, we've just shown that

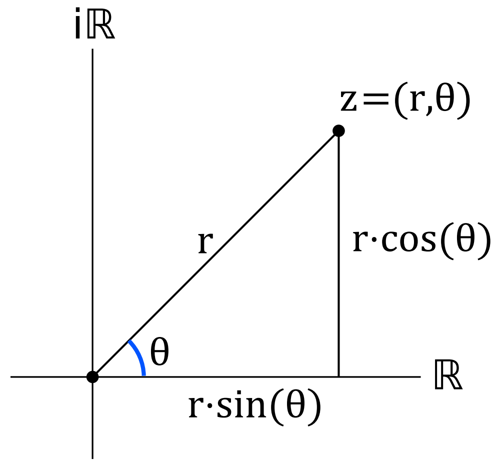
$$e^{iy} = \cos(y) + i\sin(y).$$

One famous special case of this formula is when $y = \pi$, in which case we have $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$, or

$$e^{i\pi} + 1 = 0.$$

Which is *amazing*. In one short equation, we've discovered a fundamental relation that connects five of the most fundamental mathematical constants, in a way that – without this language of power series and complex numbers – would be unfathomable to understand. Without power series, the fact that a constant related to the area of a circle (π), the square root of negative 1, the concept of exponential growth (e) and the multiplicative identity (1) can be combined to get the additive identity (0) would just seem absurd; yet, with them, we can see that this relation was inevitable from the very definitions we started from.

But that's not all! This formula (Euler's formula) isn't just useful for discovering deep fundamental relations between mathematical constants: it also gives you a way to visualize the complex plane! In specific, recall the concept of **polar coördinates**, which assigned to each nonzero point z in the plane a value $r \in \mathbb{R}^+$, denoting the distance from this point to the origin, and an angle $\theta \in [0, 2\pi)$, denoting the angle made between the positive x-axis and the line connecting z to the origin:



With this definition made, notice that any point with polar coördinates (r, θ) can be written in the plane as $(r\cos(\theta), r\sin(\theta))$. This tells us that any point with polar coördinates (r, θ) in the complex plane, specifically, can be written as $r(\cos(\theta) + i\sin(\theta))$; i.e. as $re^{i\theta}$.

This gives us what we were originally looking for: a way to visually interpret e^{x+iy}! In specific, we've shown that e^{x+iy} is just the point in the complex plane with polar coördinates (e^x, y).

2 Power Series

The motivation for **power series**, roughly speaking, is the observation that polynomials are really *quite nice*. Specifically, if I give you a polynomial, you can

- differentiate and take integrals easily,
- add and multiply polynomials together and easily express the result as another polynomial,
- find its roots,

and do most anything else that you'd ever want to do to a function! One of the only downsides to polynomials, in fact, is that there are functions that **aren't** polynomials! In specific, the very useful functions

$$\sin(x), \cos(x), \ln(x), e^x, \frac{1}{x}$$

are all not polynomials, and yet are remarkably useful/frequently occuring objects.

So: it would be nice if we could have some way of "generalizing" the idea of polynomials, so that we could describe functions like the above in some sort of polynomial-ish way – possibly, say, as polynomials of "infinite degree?" How can we do that?

The answer, as you may have guessed, is via **power series**:

Definition. A **power series** $P(x)$ centered at x_0 is just a sequence $\{a_n\}_{n=1}^{\infty}$ written in the following form:

$$P(x) = \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n.$$

Power series are almost taken around $x_0 = 0$: if x_0 is not mentioned, feel free to assume that it is 0.

The definition above says that a power series is just a fancy way of writing down a sequence. This looks like it contradicts our original idea for power series, which was that we would generalize polynomials: in other words, if I give you a power series, you quite certainly want to be able to **plug numbers into it**!

The only issue with this is that sometimes, well ... you can't:

Example. Consider the power series

$$P(x) = \sum_{n=0}^{\infty} x^n.$$

There are values of x which, when plugged into our power series $P(x)$, yield a series that fails to converge.

Proof. There are many such values of x. One example is $x = 1$, as this yields the series

$$P(x) = \sum_{n=0}^{\infty} 1,$$

which clearly fails to converge; another example is $x = -1$, which yields the series

$$P(x) = \sum_{n=0}^{\infty} (-1)^n.$$

The partial sums of this series form the sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$, which clearly fails to converge[1]. □

So: if we want to work with power series as polynomials, and not just as fancy sequences, we need to find a way to talk about where they "make sense:" in other words, we need to come up with an idea of **convergence** for power series! We do this here:

Definition. A power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

[1] Though it **wants** to converge to 1/2. Go to wikipedia and read up on Grandi's series for more information!

is said to **converge** at some value $b \in \mathbb{R}$ if and only if the series

$$\sum_{n=0}^{\infty} a_n (b - x_0)^n$$

converges. If it does, we denote this value as $P(b)$.

The following theorem, proven in lecture, is remarkably useful in telling us where power series converge:

Theorem 1. *Suppose that*

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is a power series that converges at some value $b + x_0 \in \mathbb{R}$. Then $P(x)$ actually converges on every value in the interval $(-b + x_0, b + x_0)$.

In particular, this tells us the following:

Corollary 2. *Suppose that*

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series centered at 0, and A is the set of all real numbers on which $P(x)$ converges. Then there are only three cases for A: either

1. $A = \{0\}$,
2. $A =$ *one of the four intervals* $(-b, b), [-b, b), (-b, b], [-b, b]$, *for some* $b \in \mathbb{R}$, *or*
3. $A = \mathbb{R}$.

We say that a power series $P(x)$ has ***radius of convergence*** 0 *in the first case, b in the second case, and ∞ in the third case.*

A question we could ask, given the above corollary, is the following: can we actually get all of those cases to occur? I.e. can we find power series that converge only at 0? On all of $\mathbb{R}$? On only an open interval?

To answer these questions, consider the following examples:

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} n! \cdot x^n$$

converges when $x = 0$, and diverges everywhere else.

Proof. That this series converges for $x = 0$ is trivial, as it's just the all-0 series.

To prove that it diverges whenever $x \neq 0$: pick any $x > 0$. Then the ratio test says that this series diverges if the limit

$$\lim_{n\to\infty} \frac{(n+1)!x^{n+1}}{n! \cdot x^n} = \lim_{n\to\infty} x(n+1) = +\infty$$

is > 1, which it is. So this series diverges for all $x > 0$. By applying our theorem about radii of convergence of power series, we know that our series can only converge at 0: this is because if it were to converge at any negative value $-x$, it would have to converge on all of $(-x, x)$, which is a set containing positive real numbers. $\square$

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} x^n$$

converges when $x \in (-1, 1)$, and diverges everywhere else.

Proof. Take any $x > 0$, as before, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}}{x^n} = x.$$

So the series diverges for $x > 1$ and converges for $0 \leq x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$: in our earlier discussion of power series, we proved that $P(x)$ diverged at both 1 and -1. So this power series converges on $(-1, 1)$ and diverges everywhere else. $\square$

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges when $x \in [-1, 1)$, and diverges everywhere else.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)}{x^n/n} = \lim_{n\to\infty} x \cdot \frac{n}{n+1} = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right) = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \leq x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the harmonic series (which diverges) and plugging in -1 yields the alternating harmonic series (which converges.) So this power series converges on $[-1, 1)$ and diverges everywhere else. $\square$

Example. The power series

$$P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

converges when $x \in [-1, 1]$, and diverges everywhere else.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)^2}{x^n/n^2} = \lim_{n\to\infty} x \cdot \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} x \cdot \left(1 - \frac{1}{n+1}\right)^2 = x.$$

So, again, we know that the series diverges for $x > 1$ and converges for $0 \leq x < 1$: therefore, it has radius of convergence 1, using our theorem, and converges on all of $(-1, 1)$. As for the two endpoints $x = \pm 1$, we know that plugging in 1 yields the series $\sum \frac{1}{n^2}$, which we've shown converges. Plugging in -1 yields the series $\sum \frac{(-1)^n}{n^2}$: because the series of termwise-absolute-values converges, we know that this series converges absolutely, and therefore converges.

So this power series converges on $[-1, 1]$ and diverges everywhere else. □

Example. The power series

$$P(x) = \sum_{n=0}^{\infty} 0 \cdot x^n$$

converges on all of $\mathbb{R}$.

Proof. $P(x) = 0$, for any x, which is an *exceptionally* convergent series. □

Example. The power series

$$P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges on all of $\mathbb{R}$.

Proof. Take any $x > 0$, and apply the ratio test:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

So this series converges for any $x > 0$: applying our theorem about radii of convergence tells us that this series must converge on all of $\mathbb{R}$! □

3 Complex-Valued Power Series

So: instead of considering just real-valued power series (i.e. power series where the coefficients and variables come from $\mathbb{R}$,) we can also consider complex-valued power series! In other words, power series where we allow values to come from $\mathbb{C}$! Pretty much the only tool that we'll need to study complex-valued series is the idea that **absolute convergence $\Rightarrow$ convergence**, whose statement is identical to the theorem we stated for real-valued series:

Definition. (Absolute convergence $\Rightarrow$ convergence:) Take a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers. Then, if the real-valued series

$$\sum_{n=1}^{\infty} |a_n|$$

converges, so must the series

$$\sum_{n=1}^{\infty} a_n.$$

Using this, we can talk about complex power series!

Definition. A complex power series around the point c is simply a complex valued function $f(z)$ of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$. Usually, we will study complex power series only in the case when $c=0$, in which case our power series looks like $\sum_{n=0}^{\infty} a_n z^n$.

Our definitions for complex convergence, series, and power series look fairly similar to the ones we had for real series and power series; so, we might hope that some of our theorems for power series carry through. Thankfully, many of them do, with special attention to the following theorem:

Theorem. If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is a complex power series that converges for some $z_0 \in \mathbb{C}$, then for any $a \leq |z_0|$, we have that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on the circle of radius a in $\mathbb{C}$.

This theorem has a lot of beautiful results! In particular, it tells us **why** we use the word radius in the phrase "radius of convergence:" this is because in the complex plane, the radius of convergence is an actual value r such that our power series converges for everything smaller in magnitude than r and diverges for everything in magnitude greater than r! In other words, power series on the complex plane either converge only at 0, on all of $\mathbb{C}$, or only inside some disk with some radius r (where the boundary points with magnitude r may or may not converge.)

To illustrate this picture, consider the following example:

Examples. Find the radius of convergence of the Taylor series for $\frac{1}{1+x^2}$, as considered as a complex-valued power series

Proof. Recall from our earlier lectures that

$$T\left(\frac{1}{1+x^2}\right) = 1 - x^2 + x^4 - x^6 + x^8 \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Therefore, if we look at this series at some value $r \in \mathbb{R}^+$, we can use the idea that absolute convergence $\Rightarrow$ convergence along with the ratio test to see that because

$$\lim_{n\to\infty} \frac{|r|^{2n}}{r^{2(n-1)}} = r^2,$$

whenever $r < 1$ we have that this series converges. As well, at $r=1$ our series is just

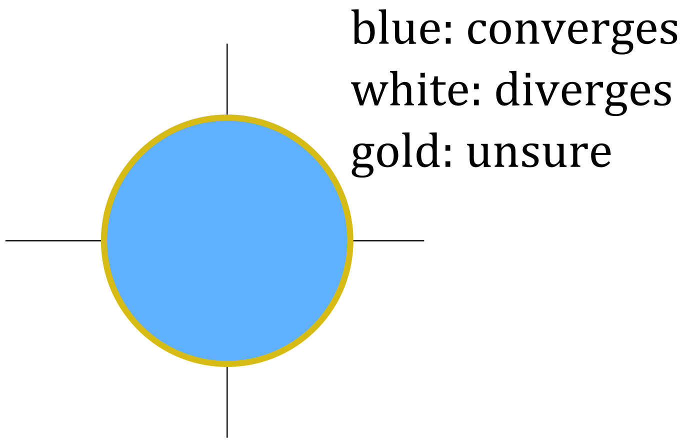
$$1 - 1 + 1 - 1 + 1 - 1 \ldots,$$

which clearly does not converge (as its partial sums oscillate between 1 and 0.) Therefore, by our theorem on radii of convergence, we know that the complex power series

$$\sum_{n=0}^{\infty} (-1)^n z^{2n}$$

must converge on all values of z with $|z| < 1$ (because for any $|z| < 1$, it converged on a value of $r > |z|$,) and diverge on all values of z with $z > 1$ (because it diverged at some value with norm 1.) In other words, its radius of convergence is 1.

To visually illustrate why this is a *radius* of convergence, we graph what we've just proven:



□

This, hopefully, should demonstrate how we attack pretty much all of these problems: to study the radius of convergence of this complex power series, all we had to do was look at its values on the real line. In general, because of our theorem on radii of convergence, this will always work! In other words, our mastery of real-valued power series will allow us to deal with complex-valued power series without much more effort.