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Lecture 7: The Tools of Integration

Week 7 *Caltech 2012*

1 The Tools of Integration

Last week, we introduced the integral, but didn't really provide any good tools for how to actually **calculate** it; as a result, integrating functions as simple as x^p was a difficult task for us. We fix this problem in this week's talks.

1.1 Definitions and Theorems

In Math 1, we've constructed a ton of definitions and theorems related to the integral. We review them here:

Definition. A set $X \subset \mathbb{R}$ is called **content zero** if the following holds: for any $\epsilon > 0$, there is some finite collection $\{I_k\}_{k=1}^n$ of closed intervals of positive length, such that

$$(1) \quad \bigcup_{k=1}^{n} I_k, \text{ the union of all of these intervals, contains } X.$$

$$(2) \quad \sum_{k=1}^{n} \text{length}(I_k) \leq \epsilon, \text{ where the length of an interval } [a,b] \text{ is } b-a.$$

In essence, sets have **content zero** if they don't take up any "space" on the real line – i.e. if we can cover them with finitely many intervals with arbitrarily small total length.

Similarly, we can define the concept of **measure zero** as a slightly weaker condition: a set X has measure 0 if for any $\epsilon > 0$, there is some collection (possibly infinite!) $\{I_k\}_{k=1}^\infty$ of closed intervals of positive length, such that

$$(1) \quad \bigcup_{k=1}^{\infty} I_k, \text{ the union of all of these intervals, contains } X.$$

$$(2) \quad \sum_{k=1}^{\infty} \text{length}(I_k) \leq \epsilon.$$

This definition roughly is the same as content zero – sets have **measure zero** if they don't take up space on the real line – but we get to use infinitely many intervals here, which makes showing that certain sets are measure zero easier.

Basically, these two definitions are ways of describing what it means for a subset of $\mathbb{R}$ to be *small*. Something we might hope for, in working with the integral, is that the integral shouldn't care about what happens on "small" sets: for example, if a function is 3 **almost** everywhere, then the area under its curve should still be considered to be 3x, provided that the places of discontinuity are on a "small" enough set.

The following theorem says, roughly, that this is always true:

Theorem. If $f(x)$ is a bounded function on the interval $[a, b]$, and the collection A of $f(x)$'s discontinuities on $[a, b]$ is a set of content zero, then the integral

$$\int_a^b f(x)dx$$

exists. Furthermore, we can redefine $f(x)$ to be whatever we want on this set of discontinuities A, (so long as we insure that the resulting function is still bounded.)

Furthermore, this theorem still holds if the set A is of measure zero, not just content zero.

This theorem is incredibly useful for showing that an integral *exists*: but what can we use to find out what it's actually equal to? The answer: the Fundamental Theorems of Calculus!

The Fundamental Theorems of Calculus, at first glance, seem like rather formidable statements: their title is set in All Caps!, their statements seem kind of ponderous, and in general they just seem like tricky things to understand and use. Luckily for us, however, these two theorems are actually really simple statements: at their heart, all that they say is that integration and derivation "undo" each other – i.e. that for continuous functions $f(x)$,

- (1st FTC) the derivative of the integral of $f(x)$ is $f(x)$, and
- (2nd FTC) the integral of the derivative of $f(x)$ is also pretty much just $f(x)$, up to a constant term C.

Put another way, the two FTC's say that integration and derivation are in some sense inverse operations to each other! (This intuitive idea should be second nature to those of you who've been through a standard calculus class before, and first encountered the idea of the integral as a kind of "antiderivative.")

We state these two theorems here:

Theorem. (The First Fundamental Theorem of Calculus:) Let $[a, b]$ be some interval. If f is a bounded and integrable function over the interval $[a, x]$ for any $x \in [a, b]$, then the function

$$A(x) := \int_a^x f(t)dt$$

exists for all $x \in [a, b]$. Furthermore, if $f(x)$ is continuous, the derivative of this function, $A'(x)$, is equal to $f(x)$.

In other words: for continuous functions $f(x)$, the integral of the derivative of $f(x)$ is just $f(x)$.

Theorem. (The Second Fundamental Theorem of Calculus:) Let $[a, b]$ be some interval. Suppose that $f(x)$ is a function that has $\varphi(x)$ as its primitive[1] on $[a, b]$; as well, suppose that $f(x)$ is bounded and integrable on $[a, b]$. Then, we have that

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a).$$

In other words: for a bounded and integrable function $f(x)$, the derivative of the integral of $f(x)$ is just $f(x)$, up to some constant term (given by $f(a)$, say.)

This idea, that integration and differentiation are kind of "opposites," motivates us to ask the following question:

Question. For derivation, we had two central tools:

- the **chain rule**: i.e. for differentiable f, g, we have $(f(g(x))' = f'(g(x)) \cdot g'(x)$.
- the **product rule**: i.e. for differentiable f, g, we have $(f(x) \cdot g(x))' = f'(x)g(x) + g'(x)f(x)$.

If we apply the fundamental theorems of calculus to these two rules, will we get a pair of "integral" theorems as well?

As it turns out: yes! Consider the following two theorems, which are direct consequences of the fundamental theorems of calculus and the chain/product rules:

Theorem. (Integration by Parts – i.e. the "integral product rule:") If f, g are a pair of C^1 functions on $[a, b]$ – i.e they have continuous derivatives on $[a, b]$ – then we have

$$\int_a^b f(x)g'(x) = f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx$$

Theorem. (Integration by Substitution – i.e. the "integral chain rule:") If f is a continuous function on $g([a, b])$ and g is a C^1 functions on $[a, b]$, then we have

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx.$$

That's a lot of theorems! We illustrate their use in the following sections, starting with the concept of measure/content zero sets, then showing how to use the fundamental theorems of calculus, and moving finally to integration by parts and by substitution.

1.2 Sets of Content and Measure Zero

Claim 1. *Any finite set has content and measure zero.*

Proof. Pick any finite set $X = \{x_1 \ldots x_n\}$, any $\epsilon > 0$, and let $I_k = (x_k - \epsilon/2n, x_k + \epsilon/2n)$. Then the union $\bigcup I_k$ contains all of the x_k's by definition; as well, because there are n total intervals and each interval has length $\leq \epsilon/n$, the total length of the I_k's is bounded by ϵ. Thus, this set has measure and content zero, as claimed. □

[1]A function $f(x)$ has $\varphi(x)$ as its **primitive** on some interval $[a, b]$ iff $\varphi'(x) = f(x)$ on all of $[a, b]$.

Claim 2. *$\mathbb{N}$ has measure zero, but not content zero.*

Proof. To see that this set does not have content zero, take any finite collection $I_1, \ldots I_n$ of intervals containing the natural numbers. There are infinitely many natural numbers and only finitely many intervals: therefore, one interval must contain infinitely many natural numbers, and thus have infinite length. Therefore it is impossible for any finite collection of intervals to both contain the natural numbers and have total length less than ϵ, for *any* finite value of ϵ.

However, if we are allowed to use infinitely many intervals, this is quite doable! Pick any $\epsilon > 0$, and let $I_k = (k - \epsilon/(2 \cdot 2^n), k + \epsilon/(2 \cdot 2^n))$. Then the union $\bigcup I_k$ contains $\mathbb{N}$ by definition; as well, we have that

$$\sum_{n=1}^{\infty} \text{length}(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon,$$

by using the geometric series $\sum_{n=1}^{\infty} 1/2^n = 1$. Therefore this set is measure zero, as claimed. $\square$

Claim 3. *$\mathbb{Q}$ has measure zero. (Because it contains $\mathbb{N}$, it does not have content zero, as we just proved that $\mathbb{N}$ does not have content zero.)*

Proof. Recall from week 2 our proof that the rational numbers are **countable**: i.e. we can pair up the rational numbers with the natural numbers, and therefore write $\mathbb{Q} = \{q_i\}_{i=1}^{\infty}$.

Then, do exactly what we did for our $\mathbb{N}$ proof: pick any $\epsilon > 0$, and let $I_k = (q_k - \epsilon/(2 \cdot 2^n), q_k + \epsilon/(2 \cdot 2^n))$. Then the union $\bigcup I_k$ contains $\{q_n\}_{n=1}^{\infty} = \mathbb{Q}$ by definition; as well, we have that

$$\sum_{n=1}^{\infty} \text{length}(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon,$$

by using the geometric series $\sum_{n=1}^{\infty} 1/2^n = 1$. Therefore this set has measure zero, as claimed. $\square$

It's worth taking a second to think about the weirdness of the above claim: the rational numbers are a set that is **dense** in the real line – they are, in a sense, everywhere. Yet, we've just shown that we can cover *all* of the rational numbers with intervals of arbitrarily small length! For example, there's a way to pick a bunch of closed intervals of positive length whose total length is just 1, and yet manage to contain *all* of the rational numbers in $\mathbb{Q}$!

Definition. We define the Cantor set C_∞ in stages, as follows:

- $C_0 = [0, 1]$.
- $C_1 = C_0$ with its middle-third removed: i.e. $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
- $C_2 = C_1$ with the middle-third of each of its intervals removed: i.e. $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.
- $\vdots$

- $C_{n+1} = C_n$ with the middle-third of each of the intervals in C_n removed.
- $\vdots$
- $C_\infty = \bigcap_{n=1}^{\infty} C_n$.

Claim 4. *The Cantor set is an uncountable set with content (and therefore measure) zero.*

Proof. Our proof goes in stages. First, notice that for every n, the set C_n is a collection of 2^n intervals each of length $1/3^n$. We can prove this by induction: our base cases are noting that C_0 is a set containing $2^0 = 1$ interval with length $1/3^0 = 1$, and C_1 is a set containing 2^1 intervals of length $1/3^1$. Inductively, if C_n is made of 2^n intervals of length $1/3^n$, then because we form C_{n+1} by cutting the middle-third out of every interval, we know that C_{n+1} has $2^n \cdot 2 = 2^{n+1}$ intervals in it, all of length $\frac{1}{3} \cdot \frac{1}{3^n} = \frac{1}{3^{n+1}}$.

Therefore, because

- C_∞ is contained within every C_n,
- every C_n is a finite collection of intervals of length $2^n/3^n$, and
- $\lim_{n\to\infty} 2^n/3^n = 0$,

we know that for any $\epsilon > 0$ we can find a n such that $2^n/3^n < \epsilon$, and therefore a finite collection C_n of intervals that contains C_∞ with total length $= 2^n/3^n < \epsilon$. So C_∞, the Cantor set, has content and measure zero.

The next claim we have is the following: the Cantor set consists of all of the real numbers in $[0, 1]$ that you can write in **ternary**[2] using only the digits 0 and 2. We only sketch our proof here, which is by induction: specifically, we claim that the numbers in C_n are precisely the numbers in $[0, 1]$ that can be written in ternary so that their first n digits don't have a 0 or 2.

Our claim is true for C_0 trivially; so, to illustrate what's going on, we look at C_1. Here, we've taken out the middle-third of the interval $[0, 1]$: in other words, we've deleted all of the numbers of the form $.1 \sim\sim\sim \ldots$ from our interval, leaving only the numbers who can be written as $.0 \sim\sim\sim \ldots$ or $.2 \sim\sim\sim \ldots$. In general, when we delete the middle-thirds of the interval C_n, we are doing precisely the same thing: we're deleting all of the numbers whose n-th digits are 1 in base 3, leaving only those which can be written using 0's and 2's in base 3.

However, if this is true, we have an obvious bijection between C_∞ and all of $[0, 1]$, given by

$$.022020220\ldots_{\text{ternary}} \mapsto .011010110\ldots_{\text{binary}},$$

the map that replaces the digit 2 with the digit 1 and interprets our number as a binary string. Therefore, because we know that $[0, 1]$ is uncountable, our set C_∞ is uncountable as well. So content zero sets can still contain an awful lot of points – uncountably infinitely many, for example! □

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[2] A number is written in **decimal**, or base 10, if when you write 21.45 you mean the number $2 \cdot 10^1 + 2 \cdot 10^0 + 4 \cdot 10^{-1} + 5 \cdot 10^{-2}$. Similarly, a number is written in **binary**, or base 2, if when you write 11.01 you mean $1 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2}$; likewise, a number is written in **ternary**, or base 3, if when you write 201.01 you mean $2 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 + 0 \cdot 3^{-1} + 1 \cdot 3^{-2}$.

1.3 Example Uses of the Fundamental Theorems of Calculus

One particular use of the Second Fundamental Theorem of Calculus is that it allows us to turn our knowledge of the derivative into knowledge about the integral: i.e. that integration is just **antidifferentiation**. To illustrate this concept, consider the following two examples:

Examples.

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1}.$$

Proof. x^p is a continuous and bounded function on $[0, b]$, for any b; furthermore, we know that

$$\left(\frac{x^{p+1}}{p+1}\right)' = \frac{p+1}{p+1}x^p = x^p, \forall x,$$

so $\frac{x^{p+1}}{p+1}$ is a primitive of x^p.

Consequently, the second fundamental theorem of calculus tells us that

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1} - \frac{0}{p+1} = \frac{b^{p+1}}{p+1},$$

as claimed. □

To get an idea of the power of the fundamental theorems of calculus, recall that proving this fact directly last week took like 8 boards worth of formulas and sums; with the fundamental theorem of calculus, this was pretty much one line.

Examples.

$$\int_a^b \cos(x)dx = \sin(b) - \sin(a)$$

Proof. Our proof here is almost identical in structure to the above proof. Note that $\cos(x)$ is a continuous and bounded function on $[a, b]$, for any a, b; furthermore, we know that

$$(\sin(x))' = \cos(x), \forall x,$$

so $\sin(x)$ is a primitive of $\cos(x)$.

Consequently, the second fundamental theorem of calculus tells us that

$$\int_0^b \cos(x)dx = \sin(b) - \sin(a),$$

as claimed. □

The first fundamental theorem of calculus has a few uses as well. One of them is dealing with integrals of the following form:

$$F(x) = \int_a^{g(x)} f(t)dt,$$

where $f(x)$ is some continuous and bounded function. How can you take the derivative of this function $F(x)$? Without the fundamental theorems of calculus, we'd be lost – simply taking the derivative of the integral itself is a difficult thing without the FTC's, and dealing with the composition of the integral with the function $g(x)$ seems inordinately difficult. Yet, with the fundamental theorems of calculus, this becomes rather simple! In fact, just let

$$H(x) = \int_a^x f(t)dt.$$

Then we have that $F(x) = H(g(x))$; consequently, the chain rule says that

$$F'(x) = H'(g(x)) \cdot g'(x).$$

Now, if we use the First Fundamental Theorem of Calculus to see that $H'(x) = f(x)$, we have

$$F'(x) = f(g(x)) \cdot g'(x);$$

something we can easily calculate!

To illustrate this method, we work two examples below:

Examples. Calculate the derivative of the function

$$F(x) = \int_0^{x^2} \sin(t)dt.$$

Proof. First, define the function $G(x)$ as

$$G(x) := \int_0^x \sin(t)dt.$$

By the fundamental theorem of calculus, we know that

$$G'(x) := \sin(x).$$

Thus, because $G(x^2) = F(x)$, we can just use the chain rule to see that

$$\begin{aligned}(F(x))' &= (G(x^2))' \\ &= 2x \cdot G'(x^2) \\ &= 2x \cdot \left(\int_0^x \sin(t)dt \right)' \Bigg|_{x^2} \\ &= 2x \cdot \sin(x^2).\end{aligned}$$

□

Examples. Calculate the derivative of the function

$$F(x) = \int_{1/x}^{x} \frac{1}{t} dt,$$

whenever $t > 0$.

Proof. First, define the function $G(x)$ as

$$G(x) := \int_{1}^{x} \frac{1}{t} dt.$$

Then, by the fundamental theorem of calculus, we have that

$$G'(x) := 1/x.$$

So: note that

$$F(x) = \int_{1/x}^{x} \frac{1}{t} dt = \int_{1}^{x} \frac{1}{t} dt - \int_{1}^{1/x} \frac{1}{t} dt = G(x) - G(1/x).$$

(Note that we defined the function G here as an integral starting at 1, not 0! This is because the integral $\int_0^x \frac{1}{t} dt$ doesn't even exist whenever x is nonzero. So, when you use linearity of your integrals to split them apart, do be careful that you're not accidentally breaking your integral into parts that don't exist!)

Then, with this expression of $F(x) = G(x) - G(1/x)$, we can just proceed by the chain rule:

$$\begin{aligned}
(F(x))' &= (G(x) - G(1/x))' \\
&= G'(x) - (-\frac{1}{x^2}) \cdot G'(1/x) \\
&= 1/x + \frac{1}{x^2} \cdot \frac{1}{1/x} \\
&= 2/x.
\end{aligned}$$

□

1.4 Examples of Integration by Parts / Integration by Substitution

Question. What's

$$\int_{1}^{2} x^2 e^x dx \ ?$$

Proof. Looking at this problem, it doesn't seem like a substitution will be terribly useful: so, let's try to use integration by parts!

How do these kinds of proofs work? Well: what we want to do is look at the quantity we're integrating (in this case, x^2e^x,) and try to divide it into two parts – a "$f(x)$"-part and a "$g'(x)$" part – such that when we apply the relation $\int f(x)g'(x) = f(x)g(x) - \int g(x)f'(x)$, our expression gets simpler!

To ensure that our expression does in fact get simpler, we want to select our $f(x)$ and $g'(x)$ such that

1. we can calculate the derivative $f'(x)$ of $f(x)$ and find a primitive $g(x)$ of $g'(x)$, so that either
2. the derivative $f'(x)$ of $f(x)$ is **simpler** than the expression $f(x)$, or
3. the integral $g(x)$ of $g'(x)$ is **simpler** than the expression $g'(x)$.

So: often, this means that you'll want to put quantities like polynomials or $\ln(x)$'s in the $f(x)$ spot, because taking derivatives of these things generally simplifies them. Conversely, things like e^x's or trig functions whose integrals you know are good choices for the integral spot, as they'll not get much more complex and their derivatives are generally no simpler.

Specifically: what should we choose here? Well, the integral of e^x is a particularly easy thing to calculate, as it's just e^x. As well, x^2 becomes much simpler after repeated derivation: consequently, we want to make the choices

$$\begin{array}{ll} f(x) = x^2 & g'(x) = e^x \\ f'(x) = 2x & g(x) = e^x, \end{array}$$

which then gives us that

$$\begin{aligned} \int_1^2 x^2 e^x dx &= f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx \\ &= x^2 e^x\Big|_1^2 - \int_1^2 2xe^x dx. \end{aligned}$$

Another integral! Motivated by the same reasons as before, we attack this integral with integration by parts as well, setting

$$\begin{array}{ll} f(x) = 2x & g'(x) = e^x \\ f'(x) = 2 & g(x) = e^x. \end{array}$$

This then tells us that

$$\begin{aligned} \int_1^2 x^2 e^x dx &= x^2 e^x\Big|_1^2 - \int_1^2 2xe^x dx \\ &= x^2 e^x\Big|_1^2 - \left(f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx \right) \\ &= x^2 e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - \int_1^2 2e^x dx \right) \\ &= x^2 e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - 2e^x\Big|_1^2 \right) \\ &= 4e^2 - e^1 - \left(4e^2 - 2e^1 - 2e^2 + 2e^1\right) \\ &= 2e^2 - e^1. \end{aligned}$$

So we're done! ☐

Question. What is

$$\int_0^2 x^2 \sin(x^3) dx \quad ?$$

Proof. How do we calculate such an integral? Direct methods seem unpromising, and using trig identities seems completely insane. What happens if we try substitution?

Well: our first question is the following: **what should we pick?** This is the only "hard" part about integration by substitution – making the right choice on what to substitute in. In most cases, what you want to do is to find the part of the integral that you don't know how to deal with – i.e. some sort of "obstruction." Then, try to make a substitution that (1) will remove that obstruction, usually such that (2) the derivative of this substitution is somewhere in your formula.

Here, for example, the term $\sin(x^3)$ is definitely an "obstruction" – we haven't developed any techniques for how to directly integrate such things. So, we make a substitution to make this simpler! In specific: Let $g(x) = x^3$. This turns our term $\sin(x^3)$ into a $\sin(g(x))$, which is much easier to deal with Also, the derivative $g'(x) = 3x^2 dx$ is (up to a constant) being multiplied by our original formula – so this substitution seems quite promising. In fact, if we calculate and use our indicated substitution, we have that

$$\begin{aligned}
\int_0^2 x^2 \sin(x^3) dx &= \int_0^2 \sin(g(x)) \cdot \frac{1}{3} \cdot g'(x) dx \\
&= \int_{0^3}^{2^3} \sin(x) dx \\
&= \frac{\sin(8)}{3} - \frac{\sin(0)}{3} \\
&= \frac{\sin(8)}{3}.
\end{aligned}$$

(Note that when we made our substitution, we also changed the bounds from $[a, b]$ to $[g(a), g(b)]$! Please, please, always change your bounds when you make a substitution!) ☐