Lecture 1: How to Count

1 What is Counting?

So: philosophically, what does it mean to count things? In other words, given some collection of sets $\{S_n\}_{n=0}^{\infty}$, what does it mean to have "counted" the elements in each S_n?

One fairly natural answer is the following: we will have counted this collection if we can come up with a closed formula, in terms of n, that will give the number of elements in n. This is often a fairly satisfying answer, as the following example shows:

Example. What is the number of ways of picking k hats out of a collection of n distinct hats? Clearly, this is just $\binom{n}{k}$, which is a beautifully simple formula! Inarguably, this is a good way to count this quantity.

Example. What are the number of ways of dividing the set $\{1, 2, 3, \ldots n\}$ into k distinct nonempty sets? We will show tomorrow in class that this quantity is

$$\sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r! \cdot (k-r)!} .$$

This formula is valid! But it's not necessarily the easiest thing to use; while it does count all of our elements, it does so in a slightly ponderous way. (Still, not too bad.)

Example. Consider the sequence formed by the following sequence of sets:

- $S_0 = \emptyset$
- $S_1 = \{1\}$
- $S_k = \{x \cup \{k\} : x \in S_{k-1} \cup S_{k-2}\}$.

Then we can recursively describe the size of each set S_k as the set $|S_{k-1}| + |S_{k-2}|$. This does enumerate the sizes of the S_k – but in an unfortunately recursive manner! Specifically, to calculate the size of S_n, we have to calculate the sizes of all of the sets from S_1 to S_{n-1} – a rather unfortunate and arduous process. So this is a somewhat "less than satisfactory" answer to our counting question.

Example. So: recall that a **partition** of an integer n is a way of writing n as a sum of a series of positive integers. One natural question we can ask is the following: given a natural number n, how can we count the number of distinct partitions of n?

The answer is the following: if $p(n)$ denotes the number of partitions of n, then

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\, \sqrt{k}\, \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right),$$

where

$$A_k(n) = \sum_{0 \le m < k\ ;\ (m,k)=1} e^{\{\pi i [s(m,k) - 2nm/k]\}}$$

and

$$s(m,k) = \frac{-1}{k} \sum_{\omega} \frac{1}{(1-\omega^m)(1-\omega)} + \frac{1}{4} - \frac{1}{4k},$$

where the sum in $s(m,k)$ is taken over all of the k-th roots of unity ω.

This is still a closed formula! – but it's absolutely awful, and nightmarish to use. A better way must exist!

So: what could be a better method of counting? Consider the following definition:

Definition. For a sequence $\{a_n\}_{n=0}^{\infty}$, the **ordinary generating function** associated to this sequence is the formal power series

$$\sum_{n=0}^{\infty} a_n \cdot x^n,$$

and the **exponential generating function** associated to this sequence is

$$\sum_{n=0}^{\infty} \frac{a_n \cdot x^n}{n!}.$$

(By a formal power series, we simply mean that we are considering these power series without concerning ourselves with questions of convergence; essentially, think of them as another way of writing down sequences that allows us to perform a large number of useful operations without much effort.)

So: given a sequence $\{a_n\}_{n=0}^{\infty}$, we claim that a generating function associated to this sequence is the "best" way of describing these elements. In other words, we're claiming that the best way to count a collection of sets is to create a generating function corresponding to the sizes of those sets!

Initially, this might seem rather farfetched; after all, writing down a sequence and attaching it to a power series doesn't seem intuitively like something that *simplifies* the sequence at all! So, consider the following example:

Example. Consider the sequence $\{a_n\}_{n=0}^{\infty}$ given by the following recurrence relation:

- $a_0 = a_1 = 1$

- $a_n = a_{n-1} + (n-1)a_{n-2}$.

What is $\{a_n\}$'s exponential generating function, $F(x)$?
Well: one rather simple-minded answer we could give is

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

But how can we write this in a nice, closed form? Well, the only thing we know about a_n is its recurrence relation; so let's try to use that! Well: one rather simple-minded answer we could give is

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \\ &= 1 + x + \sum_{n=2}^{\infty} \frac{a_n x^n}{n!} \\ &= 1 + x + \sum_{n=2}^{\infty} \frac{(a_{n-1} + (n-1)a_{n-2})x^n}{n!} \\ &= 1 + x + \sum_{n=2}^{\infty} \frac{(a_{n-1})x^n}{n!} + \sum_{n=2}^{\infty} \frac{(n-1)a_{n-2}x^n}{n!} \end{aligned}$$

So: let $A(x) = \sum_{n=2}^{\infty} \frac{(a_{n-1})x^n}{n!}$, and $B(x) = \sum_{n=2}^{\infty} \frac{(n-1)a_{n-2}x^n}{n!}$.
Then, by differentiating, we can see that

$$\begin{aligned} A'(x) &= \left(\sum_{n=2}^{\infty} \frac{(a_{n-1})x^n}{n!} \right)' \\ &= \sum_{n=2}^{\infty} \frac{(a_{n-1})nx^{n-1}}{n!} \\ &= \sum_{n=2}^{\infty} \frac{(a_{n-1})x^{n-1}}{(n-1)!} \\ &= \sum_{n=1}^{\infty} \frac{(a_n)x^n}{n!} \\ &= F(x) - 1, \end{aligned}$$

and that

$$\begin{aligned}
B'(x) &= \left(\sum_{n=2}^{\infty} \frac{(n-1)a_{n-2}x^n}{n!}\right)' \\
&= \sum_{n=2}^{\infty} \frac{(n-1)a_{n-2}nx^{n-1}}{n!} \\
&= x \cdot \sum_{n=2}^{\infty} \frac{a_{n-2}x^{n-2}}{(n-2)!} \\
&= x \cdot \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \\
&= xF(x);
\end{aligned}$$

consequently, by combining these two results we have that

$$\begin{aligned}
F'(x) &= (1+x)' + A'(x) + B'(x) \\
&= 1 + F(x) - 1 + xF(x) \\
&= (x+1)F(x).
\end{aligned}$$

This differential equation has the unique solution $e^{x+x^2/2}$ subject to the constraint that $F(0) = 1$ (which we have because $F(0) = a_0 = 1$.) So, we've found $\{a_n\}_{n=0}^{\infty}$'s generating function!

Great! So, why did we bother doing this at all? Well, because generating functions give us the following amazing list of features:

1. **Instant Recurrence Relations!** Basically, if you have a generating function, you can almost always create a recurrence relation for your variables by manipulating your power series. For example, if we take our sequence $\{a_n\}_{n=0}^{\infty}$ from the above example, we can see that

$$\begin{aligned}
F'(x) &= \left(e^{x+x^2/2}\right)' \\
&= (1+x)e^{x+x^2/2} \\
&= (1+x)\sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \qquad = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n!}.
\end{aligned}$$

But, by simply differentiating the power series directly, we also have that

$$F'(x) = \sum_{n=1}^{\infty} \frac{a_n x^{n-1}}{(n-1)!}.$$

So, as two equal formal power series have the same terms, we know that the coefficients of x^n have to agree for every n – i.e. that

$$\frac{a_{n+1}}{n!} = \frac{a_n}{n!} + \frac{a_{n-1}}{(n-1)!}$$
$$\Rightarrow \; a_{n+1} = a_n + na_{n-1}.$$

So we can easily recover recurrence relations from power series!

2. **Exact Formulas!** In addition to the above feature, we can also use generating functions to recover closed formulas for elements of our sequences! For example, if we continue to work with the $a_n's$, we can see that

$$\begin{aligned} F(x) &= e^{x+x^2/2} = e^x \cdot e^{x^2/2} \\ &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n! \cdot 2^n}\right) \\ &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{x^{2n} \cdot (2n)!}{n! \cdot 2^n \cdot (2n)!}\right) \\ &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \cdot \frac{(2k)!}{2^k \cdot k!}\right)\right) \end{aligned}$$

(if you don't see the jump between the second and third steps, try multiplying it out! i.e. look at which terms in the second step will have coefficient x^n, and sum them!)

So: generating functions will often allow us to come up with nicely closed forms for solutions, with relatively little work!

3. **Elegance!** One other feature of generating functions is that they're often much smaller, compact, and otherwise easier to work with than an exact solution. One good example is again the a_n's: their generating function, $e^{x+x^2/2}$, is far more elegant than the closed-form formula we derived above for its terms. Another, perhaps more persuasive example, is the generating function for the partition function $p(n)$ mentioned earlier! Its closed-form formula is absolutely atrocious: yet, we claim that its generating function is remarkably simple and beautiful!

To see this: notice that picking a partition of n is just a matter of choosing some amount of 1's, some amount of 2's, some amount of 3's, and so on/so forth until we've acquired enough numbers to sum up to n. So, in other words, the number of partitions of n can be thought of as the coefficient of x^n in the infinte product

$$\prod_{k=1}^{\infty} (1 + x^k + x^{2k} + x^{3k} + \ldots)$$

(as picking an x^{mk} can be thought of as being equivalent to choosing m copies of k to use in your partition.)

But we can write

$$(1 + x^k + x^{2k} + x^{3k} + \dots) = \frac{1}{1 - x^k};$$

so this means that $p(n)$'s generating function is just

$$\prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

So, they're definitely more elegant!

So: these are some of the virtues of generating functions! Not nearly all: in future classes, we'll hopefully show how generating functions can be used to create remarkably elegant identites, create asymptotic bounds, and solve lots of interesting combinatorial problems. Hopefully, however, this lecture has persuaded you that generating functions – despite their somewhat unintuitive construction! – are a remarkably useful tool with which to count things.