This problem set will continue to discuss some of the properties of trees. Problems here are (roughly, kinda) sorted from easy to difficult; use this judgement when you’re deciding which questions to study.

1. Show that a graph is a tree if and only if it has a unique spanning tree.

2. If $G$ is a graph, show that the largest acyclic subgraph of $G$ is a forest found by taking spanning trees from each of $G$’s connected components.

3. Show that any connected graph on $n$ vertices with $m$ edges has at least $m - n + 1$ cycles.

4. Show that any connected graph on $n$ vertices has one cycle if and only if it has $n$ edges. Is it true that such a graph has $m - n + 1$ cycles if and only if it has $m$ edges?

5. A graceful labeling of a graph with $E$ edges is a labeling $l(v)$ of its vertices with distinct integers from the set $\{0 \ldots E\}$, such that each edge $\{u, v\}$ is uniquely determined by the difference $|l(u) - l(v)|$.

   Show that all path-trees are graceful.

6. A caterpillar tree is a tree such that deleting all of its leaves leaves us with a single path (i.e. they kinda look like caterpillars.)

   Show that all caterpillar trees are graceful.\(^2\)

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\(^1\)A path-tree is a tree that consists of a single path.

\(^2\)The Ringel-Kotzig conjecture is the claim that all trees are graceful; graph theorists have been trying since about the 70’s to prove this claim, to no avail. Let me know if you do!