An Introduction to Graph Theory	Instructors: Marisa and Paddy
Lecture 3: Trees and Art Galleries	
Week 1 of 1	*Mathcamp 2010*

1 Glossary

cut-edge A cut-edge in a graph G is an edge whose removal increases the number of connected components of G.

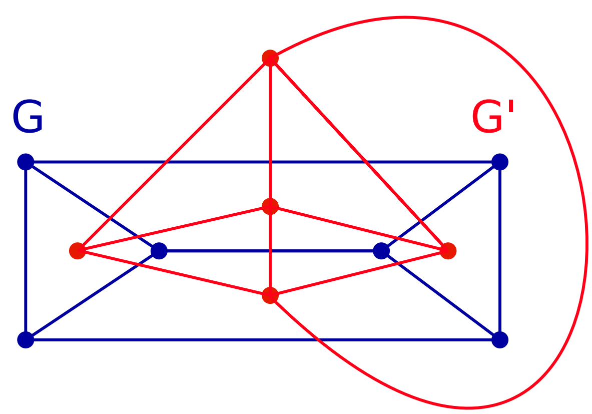
cut-vertex A cut-vertex in a graph G is a vertex whose removal increases the number of connected components of G.

spanning subgraph A spanning subgraph of G is a subgraph of G that contains every vertex of G.

planar graph A planar graph G is a graph whose edges and vertices can be drawn in $\mathbb{R}^2$ as a collection of incident points and curves, so that no two curves overlap.

face A face of a planar graph G is a region in $\mathbb{R}^2$ bounded by G's curves.

dual graph Given a planar graph G, we can define the dual graph G' as follows: let the vertices of G' be given by the collection of faces of G, and say that an edge $\{f_1, f_2\}$ exists in G' if and only if the faces f_1 and f_2 share an edge in G.



n-coloring A n-coloring of G is a way of assigning n distinct colors to the vertices of G, in such a way that no two adjacent vertices ever have the same color.

distance In a graph G, the distance $d(v, w)$ between two vertices v, w is the number of edges in the shortest path between them.

2 Cut-Edges and Spanning Trees

So: last class, we showed the following conditions were all equivalent ways to define what a tree is:

1. G is connected and has no cycles.
2. G is connected and has $n-1$ edges.
3. G has $n-1$ edges and no cycles.
4. There exists exactly one path between any two vertices in G.

Today, I'd like to add one more condition to that list:

5. G is connected, and every edge of G is a cut-edge.

To show this, we merely need to prove the following lemma:

Lemma 1 *An edge $e = \{u, v\}$ of a graph G is a cut-edge iff it doesn't belong to any cycle.*

Proof. Take any edge $e = \{u, v\}$. Remove this edge from our graph: if the graph is still connected, then there is some path from u to v not involving e; consequently, if we add e to the end of this path, we get a cycle. Thus, if e is not a cut-edge, it's involved in a cycle.

Conversely: suppose that $e = \{u, v\}$ lies in a cycle. Let P be the path from u to v that doesn't use e (i.e. go the other way around the cycle.) Pick any x, y in G; because G is connected, there's a path from x to y in G. Take this path, and edit it as follows: whenever the edge e shows up, replace this with the path P (or P traced backwards, as needed.) This then creates a walk from x to y; by deleting cycles, this walk will always become a path, and thus G is connected. So if e is involved in a cycle, it's not a cut-edge.

Corollary 2 *G is a tree if and only if it's connected and all of its edges are cut-edges.*

Proof. By the above lemma, G is connected and all of its edges are cut-edges if and only if G is connected and acyclic, which we know to be equivalent to being a tree.

Corollary 3 *Every graph G has a spanning subgraph which is a tree (a spanning subtree).*

Proof. Take a connected graph G. If G is acyclic, then G is already a tree (and thus it is a spanning subtree of itself.) If G has a cycle, delete an edge from one of G's cycles; by the above lemma, this doesn't disconnect G, and so the remaining graph is still a spanning subgraph of G. Repeat this process until an acyclic connected graph – i.e. a tree! – is created.

3 The Art Gallery Problem

Consider the following question:

Question 4 *Suppose that you have an art gallery that is shaped like some sort of n-polygon, and you want to place cameras with $360°$-viewing angles along the vertices of your polygon in such a way that the entire gallery is under surveillance. How many cameras do you need?*

So: one trivial upper bound is n. Can we do better?

Claim 5 *(Chvátal) You need at most $\lfloor n/3 \rfloor$-many cameras to guard a n-polygon.*

To prove this theorem, we need a few lemmas first:

Lemma 6 *If G is a n-polygon with $n \geq 4$, then there is some line segment formed by two of the vertices in G that lies entirely in G.*

Proof. Let v be the leftmost vertex of G. (If there is a tie, take v to be the top leftmost vertex of G.) Let u and w be v's neighbors, and examine the line segment $\overline{uw}$. If this lies entirely in G, great! Otherwise, it must cross some edge of G; consequently, there must be a vertex of G that lies inside of the triangle spanned by the three points u, v, w. Let x be the vertex furthest from the line segment $\overline{uw}$ that lies in this triangle. Then, look at the line segment $\overline{vx}$; because x is the furthest point in Δuvw from $\overline{uw}$, there can't be any edges of G that are crossed by this line segment (as one of their endpoints would necessarily be closer to v.) So $\overline{vx}$ lies entirely in G.

Corollary 7 *Any n-polygon can be divided into $n-2$-triangles.*

Proof. Using the process above, repeatedly divide our n-polygon until it's made of triangles. (by induction, the number of triangles is $n-2$.)

Lemma 8 *Let G be a polygon, T be a triangulation of G performed as above, and let T' be the dual graph to this triangulation (i.e. put a vertex in the center of every face of T , and connect two faces iff they share an edge.) If we delete the vertex corresponding to the "outer" face of T' – i.e. the one vertex of T' that doesn't correspond to a triangle – then this graph is a tree.*

Proof. Let T be our triangulated polygon. In our construction above, each of the edges of T was a diagonal of the polygon G; i.e. cutting our polygon G along one of those diagonals will always divide the polygon into two disconnnected pieces. Consequently, we know that in the dual graph T' of T, if we remove the outer face from T' every remaining edge of T' is a cut-edge! So, as we showed earlier in lecture, this means T' is a tree.

Lemma 9 *Any such triangulated polygon can be 3-colored.*

Proof. For our triangulated polygon T, take the dual graph/tree T' that we constructed above, and pick some vertex v in it. Let A_k be the collection of faces corresponding to vertices that are distance k away from v, for every k. Then, color the vertices of T as follows:

- Take the triangular face in T associated to v – i.e. the only element in A_0 – and just randomly color this triangle with the colors red, blue, and green.

- If we've colored all of the vertices corresponding to faces in A_i, for some i, then look at the faces in A_{i+1}. Each face F in this set shares exactly one edge with some triangle in A_i, and no other edges with other triangles in $\bigcup_0^{i+1} A_i$ (as its corresponding vertex has exactly one path back to v!). Thus, there is always a color available to us to choose for the third vertex of F.

- Repeat this process until every vertex in T is colored. By construction, this is a 3-coloring, as we insured that no edges were monochromatic at every step.

Corollary 10 *You need at most $\lfloor n/3 \rfloor$-many cameras to guard a n-polygon.*

Proof. By the above, create a triangulation and 3-coloring of our polygon G with the colors $\{R, G, B\}$. Without any loss of generality, assume that there are less red vertices than other colors of vertices. Then, we know that each triangle has to have one vertex of each color in it (as the triangle graph is not 2-colorable) – so putting a camera at each red vertex will suffice to cover the whole polygon!