Probabilistic Methods in Graph Theory	Instructor: Paddy
Homework 2: High Girth and Chromatic Number	
Week 1 of 1	*Mathcamp 2010*

So: 3 and 4 on yesterday's set were pretty tricky! More tricky than I remembered, even. I've added some hints here to illustrate how to attack those problems, if you want to return to them.

1 Old Problems + New Hints

1. Show that every set of $B = \{b_1, \ldots b_n\}$ of n nonzero integers contains a sum-free[1] subset of size $\geq n/3$.

   Hint: pick some prime p that's larger than twice the maximum absolute value of elements in B, and look at B modulo p – i.e., look at B in $\mathbb{Z}/p\mathbb{Z}$. Because of our clever choice of p, all of the elements in B are distinct mod p (prove this!)

   Now, look at the sets $xB := \{x \cdot b : b \in B\}$ in $\mathbb{Z}/p\mathbb{Z}$. Using the probabilstic method, show that there is some value of x such that more than a third of the elements of xB lie between $p/3$ and $2p/3$. Conclude from this (how?) that there is a sum-free subset of B of size at least $|B|/3$.

2. Let G be a graph on $n \geq 10$ vertices, and suppose that G has the following property: if we add to G any edge not in G, then the number of copies of K_{10} in G increases. Show that $|E(G)| \geq 8n - 36$.

   Hint: This was far harder than I remembered; the best outline to a solution I know follows below.

   Definition. Let S be some arbitary set, and $F = \{A_i, B_i\}_{i=1}^n$ some collection of pairs of this subset. We say that F is a (k, l)−system iff

   - $|A_i| = k$, for all i,
   - $|B_i| = l$, for all i,
   - $|A_i \cap B_i| = \emptyset$, for all i, and
   - $|A_i \cap B_j| \neq \emptyset$, for all distinct $i \neq j$.

   Proposition 1 *If $F = \{A_i, B_i\}_{i=1}^n$ is a (k, l)-system, then $n \leq \binom{k+l}{l}$.*

   Proof. (Sketch): let $X = \bigcup_{i=1}^n (A_i \cup B_i)$, and consider a random ordering π of X's elements. For each i, let X_i be the event that all of A_i's elements preceed B_i's elements in this order. Then, we have the following (which you can show!)

[1] A subset of $\mathbb{R}$ is called sum-free if adding any two elements in the subset will never give you an element of the subset.

- $Pr(X_i) = 1/\binom{k+l}{l}$.
- All of the events X_i are pairwise disjoint; i.e. there is no way for two of the X_i's to occur simultaneously.

Thus, we have that

$$1 \geq Pr\left(\bigcup_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Pr(X_i) = n/\binom{k+l}{l};$$

thus, we have that $n \leq \binom{k+l}{l}$ as claimed.

Proposition 2 *Define the following sets:*

- *For our graph G, let $\bar{E}$ be the collection of non-edges of our graph G.*
- *For every edge e in E, pick some K_{10} formed by adding e: let A_e be the collection of vertices in G that are **not** involved in this K_{10}.*
- *For every edge e in E, define B_e as the endpoints of the edge e.*

We claim that this forms a $(n-10, 2)$-system.

Corollary 3 *By combining our above propositions, $|\bar{E}| \leq \binom{n-8}{2}$; consequently, we have that*

$$|E| = \binom{n}{2} - |\bar{E}| \geq \binom{n}{2} - \binom{n-8}{2} = 8n - 36.$$

2 New Problems

3. Construct (i.e. don't use probabilistic methods!) a triangle-free graph with arbitarily large chromatic number.

   Hint: induct! i.e. let G_2 be a graph with a single edge. Construct G_{n+1} from G_n as follows:

   - At first, let G_{n+1} be the union of a copy of G_n on the vertex set V, along with V' new vertices (where $|V| = |V'|$,) and one more new vertex x.
   - For every vertex $v' \in V'$ that's a copy of some $v \in V$, add edges from v' to every $w \in V$ such that $(v, w) \in G_n$.
   - Connect x to every vertex of V'.

   Show that this works!

4. As above, but with girth ≥ 6!