Probabilistic Methods in Graph Theory	Instructor: Paddy
Lecture 1: Probabilistic Methods: An Introduction	
Week 1 of 1	*Mathcamp 2010*

1 Glossary

Ramsey number The Ramsey number $R(k, l)$ is the smallest number n such that any red-blue two-coloring of K_n's edges will always create a red K_k or a blue K_l.

Finite sample space A finite sample space is just some finite set Ω.

Probability function Given a finite sample space Ω, a probability function Pr on Ω is just a map $Pr : \Omega \to [0, 1]$ with the property that

$$\sum_{\omega in \Omega} Pr(\omega) = 1.$$

Finite probability space A pair (Ω, Pr), where Ω is a finite sample space and Pr is a probability function.

Uniform distribution A pair (Ω, Pr), where Ω is a finite sample space and Pr is the probability function given by $\Pr(\omega) = 1/|\Omega|$, for every $\omega \in \Omega$.

Event An event A is just some subset of a finite sample space.

Random variable A random variable X on some finite sample space Ω is just a map from Ω to $\mathbb{R}$.

Expectation The expectation of a random variable X is the integral of X over Ω. For finite spaces, this is just the sum

$$\sum_{\omega in \Omega} Pr(\omega) \cdot X(\omega).$$

2 Example 1: Ramsey Numbers

The probabilistic method in combinatorics first arose in 1947, when Erdös used it to prove the following claim:

Theorem 1 $R(k, k) > \lfloor 2^{k/2} \rfloor$.

Proof. Fix some value of n, and consider a random uniformly-chosen 2-coloring of K_n's edges: in other words, let us work in the probability space $(\Omega, Pr) =$ (all 2-colorings of K_n's edges, $Pr(\omega) = 1/2^{\binom{n}{2}}$.)

For some fixed set R of k vertices in $V(K_n)$, let A_R be the event that the induced subgraph on R is monochrome. Then, we have that

$$Pr(A_R) = 2 \cdot \left(2^{\binom{n}{2}-\binom{k}{2}}\right)/2^{\binom{n}{2}} = 2^{1-\binom{k}{2}}.$$

Thus, we have that the probability of at least one of the A_R's occuring is bounded by

$$Pr(\bigcup_{|R|=k} A_R) \leq \sum_{R\subset\Omega,|R|=k} Pr(A_R) = \binom{n}{k}2^{1-\binom{k}{2}}.$$

If we can show that $\binom{n}{k}2^{1-\binom{k}{2}}$ is less that 1, then we know that with nonzero probability there will be some 2-coloring $\omega \in \Omega$ in which none of the A_R's occur! In other words, we know that there is a 2-coloring of K_n that avoids both a red and a blue K_k.

Solving, we see that

$$\binom{n}{k}2^{1-\binom{k}{2}} < \frac{n^k}{k!} \cdot 2^{1+(k/2)-(k^2/2)} = \frac{2^{1+k/2}}{k!} \cdot \frac{n^k}{2^{k^2/2}} < 1$$

whenever $n = \lfloor 2^{k/2} \rfloor, k \geq 3$. So we're done!

So: why did we do this? In other words, what did using probabilistic methods gain us?

The answer, essentially, is that the probabilistic method allows us to work with graphs that are both *large* and *unstructured*! When using constructive methods, we can rarely (if at all) do this! I.e.:

- If you're trying to construct a large graph by gluing together pieces of smaller graphs, you are almost always inducing a lot of structure into your larger graph; consequently, your construction will usually be a highly atypical graph! For example, try constructing a graph of both girth and chromatic number greater than 6 – you'll quickly find that it's stunningly difficult to avoid introducing structure in any building method that won't create small cycles or small chromatic numbers. Yet, using the probabilistic method we can easily show that there are graphs of arbitrarily high girth and chromatic number! – in fact, that almost all sufficiently large graphs are such things.

- Conversely, suppose that you're trying to avoid such problems, and have decided to simply check by hand all of the cases for some reasonably small number of vertices – say, 20. But there are $2^{\binom{20}{2}} = 2^{190} \approx 1.5 * 10^{57}$ such graphs! Even with stunningly powerful supercomputers, there's no hope. Yet, with the probabilistic method, we will routinely create counterexamples with $> 10^{10}$ vertices in them! – things we could never hope to find in any deterministic search.

3 Example 2: Splitting Graphs

We close here with one last example of the probabilistic method:

Theorem 2 *If G is a graph, then G contains a bipartite subgraph with at least $E/2$ edges.*

Proof. Pick a subset of G's vertices, T, uniformly at random (i.e. select T by flipping a coin for each of G's vertices, and placing vertices in T iff our coin comes up heads.) Let $B = V(G) \setminus T$.

Call an edge $\{x, y\}$ of $E(G)$ **crossing** iff exactly one of x, y lie in T, and let X be the random variable defined by

$$X(T) = \text{number of crossing edges for } T.$$

Then, we have that

$$X(T) = \sum X_{x,y}(T),$$

where $X_{x,y}(T)$ is the 0-1 random variable defined by $X_{x,y}(T) = 1$ if $\{x, y\}$ is an edge of G that's crossing, and 0 otherwise.

The expectation $\mathbb{E}(X_{x,y})$ is clearly $1/2$, because we chose x and y to be in T at random. Thus, by the linearity of expectation, we have that

$$\mathbb{E}(X) = \sum \mathbb{E}(X_{x,y}) = E/2.$$

so the expected number of crossing edges for a random subset of G is $E/2$. Thus, there must be some $T \subset V(G)$ such that $X(T) \geq E/2$; taking the collection of crossing edges this set creates then gives us a bipartite graph (B, T) with $\geq E/2$ edges in it.