Latin Squares | Instructor: Paddy

# Homework 1: Latin Squares!

*Week 4* | *Mathcamp 2010*

1. Let $L(n)$ denote the number of $n \times n$ latin squares, and $l(n)$ denote the number of $n \times n$ latin squares where the first row and column are both of the form $[1, \ldots n]$. Prove the following formula:

$$L(n) = (n)!(n-1)!l(n).$$

2. How many non-equivalent latin squares of order 4 are there?

3. If you haven't yet, prove Hall's marriage theorem: i.e.

**Theorem 1** *Suppose that $G = (A, B)$ is a bipartite graph that satisfies **Hall's property**:*

$$(\ddagger): \qquad \forall H \subset A \text{ or } H \subset B, |N(H)| \geq |H|.$$

*Then $G$ has a 1-factor.*

4. Prove the following lemma from class:

**Lemma 2** *If $A$ is a $n \times n$ integral matrix, then for any $d$ there are matrices $B_1, \ldots B_d$ such that*

$$A = B_1 + \ldots + B_d,$$

*where the matrices $B_i$ are all integral $n \times n$ matrices that have the same entries, row and column sums, and sum over all entries as $\frac{1}{d}A$, up to rounding up or down.*

5. Show that a $n \times n$ latin square is equivalent to a 1-factorization of $K_{n,n}$.

6. Show that a $n \times n$ latin square is equivalent to a triangulation of $K_{n,n,n}$.

7. Show that the multiplication table of any group $G$ on $n$ elements forms a latin square. Are there latin squares that don't arise in this way?